Time-consistent stopping under decreasing impatience*

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Under non-exponential discounting, we develop a dynamic theory for stopping problems in continuous time. Our framework particularly covers discount functions that induce decreasing impatience. Due to the inherent time inconsistency, the stopping problem is presented as an inter-temporal game among a continuum of non-cooperative players. We look for equilibrium stopping policies, which are formulated as fixed points of an operator. Under appropriate conditions, we show that fixed-point iterations converge to equilibrium stopping policies. This in particular provides an explicit connection between optimal stopping times in classical stopping literature and equilibrium stopping policies under current game-theoretic setting. This connection is new in the literature of time-inconsistent problems, and it corresponds to increasing levels of strategic reasoning. Our theory is illustrated in a real option pricing model.

Keywords: time-inconsistency; stopping; hyperbolic discounting; decreasing impatience; sequential game; sub-game perfect equilibrium

JEL: C61; D81; D90; G02

2010 Mathematics Subject Classification: 60G40; 91B06

1 Introduction

In this article, we build a general theory for stopping problems under time-inconsistent (or dynamically-inconsistent) preferences. Time inconsistency of stopping decisions is known to exist in casino gambling and financial planning. A general treatment, however, has not been proposed so far; see the conclusion of Ebert et al. (2015). Here, we focus on the stopping of a diffusion process under infinite horizon, in order to maximize one's expected discounted payoff. Time inconsistency comes from the consideration of non-exponential discount functions.

1.1 Literature

The study of time inconsistency goes back to the seminal work of Stroz (1955), where the author identifies three different kinds of individuals—the *naive*, the *precommitted*, and the *sophisticated*—in the face of time inconsistency. A *naive* agent constantly changes her choice of strategies, in response to her constantly-changing preferences, without realizing the problem of time inconsistency. A *precommitted* agent simply forces herself to stick with the original optimal strategy, with no justifiable reason. By contrast, a *sophisticated* agent works on *consistent planning*: she takes into account the possible change of preferences in the first place, and aims

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to find a strategy that once enforced over time, none of her future selves would want to deviate from it. While a sophisticated strategy can be defined through backward iteration in finite discrete time (Hayashi (2009)), it has been unclear in the literature how to formulate this notion precisely in either infinite discrete time or continuous time.

A recent literature, particularly in Mathematical Finance, has emerged in response to this issue. This includes portfolio choice under non-exponential discounting in Ekeland and Lazrak (2006), Ekeland and Pirvu (2008), and Ekeland, Mbodji, and Pirvu (2012), mean-variance portfolio optimization in Björk, Murgoci, and Zhou (2014) and the related stochastic linear-quadratic control in Hu, Jin, and Zhou (2012, 2015), utility indifference pricing with risk aversion varying over different regimes in Pirvu and Zhang (2013), and investment-consumption problem with state-dependent risk aversion in Dong and Sircar (2014). In all these different time-inconsistent control problems, sub-game perfect Nash equilibrium emerges as the proper formulation for sophisticated strategies. In a nutshell, a sophisticated agent continuously plays an inter-temporal game among herself today and all her future selves. Built upon this equilibrium idea, a fairly general theory of Markovian time-inconsistent stochastic control takes shape in Björk and Murgoci (2014b,a). Research along this line can also be found in Yong (2012).

There is, however, no equivalent development for stopping problems. Under non-exponential discounting, Grenadier and Wang (2007) study optimal investment time in a real options model. Since an agent is only allowed to update her time preferences at discrete moments, their analysis relies on standard backward iteration and does not shed much light on the general continuous-time formulation. Barberis (2012) investigates optimal exit strategies in casino gambling. He underlines the inherent time inconsistency, and analyses in detail the naive and pre-committed strategies, whereas the sophisticated one is not mentioned. Xu and Zhou (2013) characterize optimal stopping times under probability distortion. They remark on the involved time inconsistency, but do not attempt to resolve it.

1.2 Contributions

Our main contribution is the precise formulation of sophisticated strategies for stopping problems. We introduce an operator Θ in (3.5), which describes the behavior of a sophisticated agent. Specifically, for any given stopping policy τ, the sophisticated agent improves τ through one level of strategic reasoning, i.e. anticipating the behavior of her future selves. Sophisticated stopping policies are then defined as fixed points of Θ, which naturally connects to the concept of sub-game perfect Nash equilibrium; see Definition 3.2 and the discussion underneath.

When the objective function is taken as an expected discounted payoff, we perform fixed-point iterations to search for sophisticated (equilibrium) stopping policies. That is, for any stopping policy τ, we apply the operator Θ onto τ repetitively, and expect convergence to some fixed point of Θ. Here, two crucial conditions are needed. First, we assume that the discount function $\delta : \mathbb{R}_+ \mapsto [0,1]$ is log sub-additive, i.e., $\delta(s)\delta(t) \leq \delta(s+t)$ for all $s,t \geq 0$. This condition in particular captures *decreasing impatience*, an acknowledged feature of empirical discounting in Behavioral Economics; see e.g. Thaler (1981), Loewenstein and Thaler (1989), and Loewenstein and Prelec (1992). Hyperbolic and quasi-hyperbolic discounting are special cases under our framework; see Assumption 3.1 and the discussion below it for details. The second condition, Assumption 3.2, is imposed on the initial stopping policy τ the fixed-point iteration starts with. These two conditions together imply monotonicity of the iteration $\{\Theta^n\tau\}_{n\in\mathbb{N}}$ (Proposition 3.3). The iteration, as a result, converges, and we show that the limiting stopping policy is indeed an equilibrium in Theorem 3.1, which is the main result of our paper.

An interesting observation is that the stopping strategy of a naive agent, denoted by $\widetilde{\tau}$, satisfies Assumption 3.2. Our main result then implies $\widetilde{\tau}_0 := \lim_{n\to\infty} \Theta^n\widetilde{\tau}$ is an equilibrium stopping policy. This relation states an explicit connection between the (irrational) naive behavior and the (fully rational) sophisticated one, through increasing levels of strategic reasoning of the agent. Such a link is new in the literature; see the discussion below Corollary 3.1 for

details. Notably, examples show that equilibrium may be found after only *finite* applications of Θ on the naive strategy.

As an illustration of our theory, we introduce in Section 4 a simple model of irreversible investment planning under a real options framework, where the value process is a one-dimensional Bessel process and the discount function is hyperbolic. We derive an explicit formula for the naive stopping policy, and provide characterizations for equilibrium policies and all intermediary policies obtained through repetitive application of Θ.

The paper is organized as follows. In Section 2, we introduce the setup of our model, and demonstrate time inconsistency in stopping problems through examples. In Section 3, we first examine a sophisticated agent's behavior from a game-theoretic perspective. Based on our observations, we precisely formulate the concept of (sub-game perfect Nash) equilibrium for stopping problems in continuous time. Next, we propose to find these equilibriums via fixed-point iterations, and establish the required convergence result. Section 4 thoroughly studies the real option model under hyperbolic discounting as an illustration of our theory. Most of the proofs are delegated to the appendices.

2 Preliminaries and Motivation

2.1 The Model

Consider the canonical space $\Omega := \{\omega \in C([0,\infty);\mathbb{R}^d) : \omega_0 = 0\}$. Let $\{W_t\}_{t\geq 0}$ be the coordinate mapping process $W_t(\omega) = \omega_t$, and $\mathbb{F}^W = \{\mathcal{F}_s^W\}_{s\geq 0}$ be the natural filtration generated by W, i.e. $\mathcal{F}_s^W = \sigma(W_u : 0 \leq u \leq s)$. Let $\mathbb{P}$ be the Wiener measure on $(\Omega, \mathcal{F}_\infty^W)$, where $\mathcal{F}_\infty^W := \bigcup_{s\geq 0} \mathcal{F}_s^W$. For each $t \geq 0$, we introduce the filtration $\mathbb{F}^{t,W} = \{\mathcal{F}_s^{t,W}\}_{s\geq 0}$ with

$$\mathcal{F}_s^{t,W} = \sigma(W_{u\vee t} - W_t : 0 \leq u \leq s), \tag{2.1}$$

and let $\mathbb{F}^t = \{\mathcal{F}_s^t\}_{s\geq 0}$ be the $\mathbb{P}$-augmentation of $\mathbb{F}^{t,W}$. We denote by $\mathcal{T}_t$ the collection of all $\mathbb{F}^t$-stopping times τ with $\tau \geq t$ a.s. For the case where $t = 0$, we simply write $\mathbb{F}^0 = \{\mathcal{F}_s^0\}_{s\geq 0}$ as $\mathbb{F}_s = \{\mathcal{F}_s\}_{s\geq 0}$, and $\mathcal{T}_0$ as $\mathcal{T}$.

Remark 2.1. *In view of (2.1), for any $0 \leq s \leq t$, $\mathcal{F}_s^t$ is the σ-algebra generated by only the $\mathbb{P}$-negligible sets. Moreover, for any $s, t \geq 0$, $\mathcal{F}_s^t$-measurable random variables are independent of $\mathcal{F}_t$; see Bouchard and Touzi (2011), particularly Remark 2.1 therein, for a similar set-up.*

Now, fix $t \geq 0$ and $\omega \in \Omega$. We define the concatenation of ω and $\tilde\omega \in \Omega$ at time t by

$$(\omega \otimes_t \tilde\omega)_s := \omega_s 1_{[0,t)}(s) + [\tilde\omega_s - (\tilde\omega_t - \omega_t)]1_{[t,\infty)}(s), \quad s \geq 0.$$

By construction, $\omega \otimes_t \tilde\omega$ belongs to Ω. Furthermore, for any $\mathcal{F}_\infty$-measurable random variable $\xi : \Omega \mapsto \mathbb{R}$, we define the shifted random variable $[\xi]_{t,\omega} : \Omega \mapsto \mathbb{R}$, which is $\mathcal{F}_\infty^t$-measurable, by

$$[\xi]_{t,\omega}(\tilde\omega) := \xi(\omega \otimes_t \tilde\omega), \quad \forall \tilde\omega \in \Omega.$$

For any $\tau \in \mathcal{T}$, we will simply write $\omega \otimes_{\tau(\omega)} \tilde\omega$ as $\omega \otimes_\tau \tilde\omega$, and $[\xi]_{\tau(\omega),\omega}$ as $[\xi]_{\tau,\omega}$. Note that the concatenation of paths in Ω and shifted random variables were introduced in Nutz (2013). Bayraktar and Huang (2013, Appendix A) contains a detailed analysis of the properties of shifted random variables; in particular, Proposition A.1 therein shows that for any $\tau \in \mathcal{T}$ and $\mathcal{F}_\infty$-random variable ξ with $\mathbb{E}[|\xi|] < \infty$,

$$\mathbb{E}[\xi \mid \mathcal{F}_\theta](\omega) = \mathbb{E}\left[[\xi]_{\theta,\omega}\right] \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{2.2}$$

Consider the space $\mathbb{X} := [0,\infty) \times \mathbb{R}^d$, equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{X})$. Let $b : \mathbb{X} \mapsto \mathbb{R}$ and $\sigma : \mathbb{X} \mapsto \mathbb{R}$ satisfy Lipschitz and linear growth conditions: there is $K > 0$ such that for any $t \in [0,\infty)$ and $x, y \in \mathbb{R}^d$,

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|, \quad |b(t,x)| + |\sigma(t,x)| \leq K(1+|x|).$$

Then, for any $\tau \in \mathcal{T}$ and $\mathbb{R}^d$-valued $\mathcal{F}_\tau$-measurable random variable ξ with $\mathbb{E}[|\xi|^2] < \infty$, the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \quad \text{for } t \geq \tau, \qquad \text{with } X_\tau = \xi \text{ a.s.} \tag{2.3}$$

admits a unique strong solution, which will be denoted by $X^{\tau,\xi}$.

2.2 Classical Optimal Stopping

We introduce a *payoff function* $g : \mathbb{R}^d \mapsto \mathbb{R}$, assumed to be nonnegative and continuous; a *discount function* $\delta : \mathbb{R}_+ \mapsto [0, 1]$, assumed to be continuous, decreasing, and satisfy $\delta(0) = 1$. Moreover, we assume that

$$\mathbb{E}\left[\sup_{t \leq s \leq \infty} \delta(s - t)g(X_s^{t,x})\right] < \infty, \quad \forall (t, x) \in \mathbb{X}, \tag{2.4}$$

where we interpret $\delta(\infty - t)g(X_\infty^{t,x}) := \lim_{s\to\infty} \delta(s - t)g(X_s^{t,x})$; this is in line with the setup in Karatzas and Shreve (1998, Appendix D). Classical literature of optimal stopping (see e.g. Karatzas and Shreve (1998, Appendix D), Peskir and Shiryaev (2006, Chapter I.2), and the abundant references therein) is concerned with the problem: given initial time and state $(t, x) \in \mathbb{X}$, can one find a $\tau \in \mathcal{T}_t$ such that her expected discounted payoff

$$J(t, x; \tau) := \mathbb{E}\left[\delta(\tau - t)g(X_\tau^{t,x})\right] \tag{2.5}$$

can be maximized? The associated value function

$$v(t, x) := \sup_{\tau \in \mathcal{T}_t} J(t, x; \tau) \tag{2.6}$$

has been widely studied, and the existence of an optimal stopping time is affirmative. The following is a standard result taken from Karatzas and Shreve (1998, Appendix D) and Peskir and Shiryaev (2006, Chapter I.2).

Proposition 2.1. *For any $(t, x) \in \mathbb{X}$, let $\{Z_s^{t,x}\}_{s\geq t}$ be a right-continuous process such that*

$$Z_s^{t,x} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t,\ \tau\geq s} \mathbb{E}[\delta(\tau - t)g(X_\tau^{t,x}) \mid \mathcal{F}_s] \quad \text{a.s.} \quad \forall s \geq t, \tag{2.7}$$

and define $\widetilde{\tau}(t, x) \in \mathcal{T}_t$ by

$$\widetilde{\tau}(t, x) := \inf\left\{s \geq t \ :\ \delta(s - t)g(X_s^{t,x}) = Z_s^{t,x}\right\}. \tag{2.8}$$

Then, $\widetilde{\tau}(t, x)$ is an optimal stopping time of (2.6), i.e.

$$J(t, x; \widetilde{\tau}(t, x)) = \sup_{\tau\in\mathcal{T}_t} J(t, x; \tau). \tag{2.9}$$

Moreover, $\widetilde{\tau}(t, x)$ is the smallest, if not unique, optimal stopping time.

Throughout this paper, we will constantly use the notation

$$\mathbb{E}^{t,x}[\delta(\tau - t)g(X_\tau)] = \mathbb{E}[\delta(\tau - t)g(X_\tau^{t,x})], \quad \forall (t, x) \in \mathbb{X} \text{ and } \tau \in \mathcal{T}_t.$$

By (2.2), we observe that for any $0 \leq t \leq s$ and $\tau \in \mathcal{T}_t$ with $\tau \geq s$,

$$\mathbb{E}[\delta(\tau - t)g(X_\tau^{t,x}) \mid \mathcal{F}_s](\omega) = \mathbb{E}\left[\left[\delta(\tau - t)g(X_\tau^{t,x})\right]_{s,\omega}\right] = \mathbb{E}^{s,X_s^{t,x}(\omega)}[\delta([\tau]_{s,\omega} - t)g(X_{[\tau]_{s,\omega}})].$$

Since $[\tau]_{s,\omega} \in \mathcal{T}_s$, we conclude that (2.7) can be rewritten as

$$Z_s^{t,x} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s} \mathbb{E}^{s,X_s^{t,x}(\omega)}[\delta(\tau - t)g(X_\tau)] \quad \text{a.s.} \tag{2.10}$$

2.3 Time Inconsistency

The classical optimal stopping problem (2.6) is *static*, in the sense that it involves only the preference of the agent at time t, formulated as the expected discounted payoff in (2.5). Following Stroz (1955), a naive agent simply solves the classical optimal stopping problem (2.6) repeatedly at every moment as time passes by. That is, given initial $(t,x)\in\mathbb{X}$, the agent aims to solve

$$\sup_{\tau\in\mathcal{T}_s} J(s, X_s^{t,x};\tau) \quad \text{at every moment } s\geq t. \tag{2.11}$$

By Proposition 2.1, the agent at time s intends to employ the stopping time $\widetilde{\tau}(s, X_s^{t,x})\in\mathcal{T}_s$, for all $s\geq t$. This naturally raises the question of whether optimal stopping times obtained at different moments, $\widetilde{\tau}(t,x)$ and $\widetilde{\tau}(t', X_{t'}^{t,x})$ with $t'>t$, are consistent with each other.

Definition 2.1 (Time Consistency). *We say the problem (2.11) is time-consistent if the following holds: for any $(t,x)\in\mathbb{X}$, $s>t$, and a.e. $\omega\in\Omega$, if $\widetilde{\tau}(t,x)(\omega)\geq s$, then*

$$[\widetilde{\tau}(t,x)]_{s,\omega}(\tilde{\omega}) = \widetilde{\tau}(s, X_s^{t,x}(\omega))(\tilde{\omega}) \quad \textit{for a.e. } \tilde{\omega}\in\Omega.$$

We say the problem (2.11) is time-inconsistent if the above does not hold.

In the classical literature of Mathematical Finance, the discount function usually takes the form $\delta(s)=e^{-\rho s}$ for some $\rho\geq 0$. This already guarantees time consistency of (2.11). To see this, first observe the identity

$$\delta(s)\delta(t)=\delta(s+t) \quad \forall s,t\geq 0. \tag{2.12}$$

Fix $(t,x)\in\mathbb{X}$ and pick $t'>t$ such that $\mathbb{P}[\widetilde{\tau}(t,x)\geq t']>0$. For a.e. $\omega\in\{\widetilde{\tau}(t,x)\geq t'\}$, set $y:=X_{t'}^{t,x}(\omega)$. We observe from (2.8), (2.10), and $X_s^{t,x}(\omega\otimes_{t'}\tilde{\omega}) = X_s^{t',y}(\tilde{\omega})$ that

$$[\widetilde{\tau}(t,x)]_{t',\omega} = \inf\left\{s\geq t' : \delta(s-t)g(X_s^{t',y}) \geq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s} \mathbb{E}^{s,X_s^{t',y}}[\delta(\tau-t)g(X_\tau)]\right\}, \tag{2.13}$$

$$\widetilde{\tau}(t', X_{t'}^{t,x}(\omega)) = \inf\left\{s\geq t' : \delta(s-t')g(X_s^{t',y}) \geq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s} \mathbb{E}^{s,X_s^{t',y}}[\delta(\tau-t')g(X_\tau)]\right\}. \tag{2.14}$$

By (2.12), we obtain

$$[\widetilde{\tau}(t,x)]_{t',\omega} = \widetilde{\tau}(t', X_{t'}^{t,x}(\omega)) = \inf\left\{s\geq t' : g(X_s^{t',y}) \geq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s} \mathbb{E}^{s,X_s^{t',y}}[\delta(\tau-s)g(X_\tau)]\right\}.$$

With non-exponential discount functions, the problem (2.11) is time-inconsistent: since the identity (2.12) does not necessarily hold, (2.13) and (2.14) do not coincide in general.

Example 2.1 (Smoking Cessation). *Suppose a smoker has a fixed lifetime $T>0$. For each $(t,x)\in[0,T]\times\mathbb{R}_+$, consider the deterministic cost process $X_s^{t,x}=xe^{\frac{1}{2}(s-t)}$, for $s\in[t,T]$. The smoker can either (i) quit smoking at some time $s<T$ (with cost X_s) and then die peacefully at time T (with no cost), or (ii) never quit smoking during his lifetime (thus incurring no cost) but eventually die painfully at time T (with cost X_T). With hyperbolic discount function $\delta(s):=\frac{1}{1+s}$ for $s\geq 0$, the classical optimal stopping problem (2.6) of minimizing the cost becomes*

$$\inf_{s\in[t,T]}\delta(s-t)X_s^{t,x} = \inf_{s\in[t,T]}\frac{xe^{\frac{1}{2}(s-t)}}{1+(s-t)}.$$

By basic Calculus, the optimal stopping time $\widetilde{\tau}(t,x)$ is given by

$$\widetilde{\tau}(t,x) = \begin{cases} t+1 & \text{if } t<T-1, \\ T & \text{if } t\geq T-1. \end{cases} \tag{2.15}$$

Time inconsistency can be easily observed. Here, it particularly illustrates the procrastination behavior: the smoker never quits smoking.

Example 2.2 (Real Option Model). *Suppose $\{X_t\}_{t\geq 0}$ is a standard Brownian motion. Consider the payoff function $g(x) := |x|$ for $x \in \mathbb{R}$ and the hyperbolic discount function $\delta(s) := \frac{1}{1+s}$ for $s \geq 0$. Given initial $(t,x) \in \mathbb{X}$, the classical optimal stopping problem (2.6) becomes*

$$\sup_{\tau\in\mathcal{T}_t} \mathbb{E}^{t,x}\left[\frac{|X_\tau|}{1+(\tau-t)}\right],$$

which can be viewed as a real options valuation for finding the optimal time to initiate or abandon an investment project. In Proposition 4.1 below, we show that $\widetilde{\tau}(t,x)$ in (2.8) has the formula

$$\widetilde{\tau}(t,x) = \inf\left\{s \geq t : |X_s^{t,x}| \geq \sqrt{1+(s-t)}\right\}. \tag{2.16}$$

The free boundary $s \mapsto \sqrt{1+(s-t)}$ is unusual in its dependence on initial time t. As time passes by (i.e. t increases), the free boundary keeps changing, as shown in Figure 1. We clearly observe time inconsistency: $[\widetilde{\tau}(t,x)]_{t',\omega} \neq \widetilde{\tau}(t', X_{t'}^{t,x}(\omega))$, for any $t' > t$, as they correspond to different free boundaries.

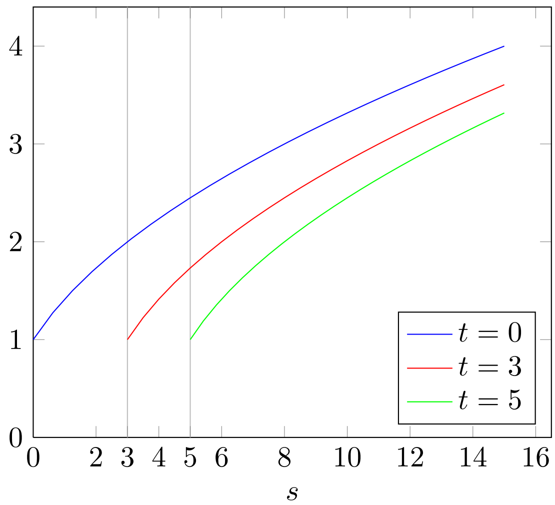


Figure 1: The free boundary $s \mapsto \sqrt{1+(s-t)}$ with different initial times t.

As proposed in Stroz (1955), to deal with time inconsistency, we need to employ a strategy that is either *pre-committed* or *sophisticated*. A pre-committed agent picks the optimal stopping time $\widetilde{\tau}(t,x)$ in (2.8) at initial time t, and forces her future selves to follow $\widetilde{\tau}(t,x)$ through some commitment mechanism (e.g. a contract). On the other hand, a sophisticated agent anticipates in the first place future preference updating, and aims to find a stopping strategy that once being enforced, none of her future selves would want to deviate from it. To precisely formulate sophisticated stopping strategies has been a challenge in continuous time. The following section focuses on resolving this.

3 Equilibrium Stopping Policies

3.1 Objective of a Sophisticated Agent

Since one may re-evaluate and change her choice of stopping times over time, an agent's stopping strategy is not a single stopping time, but a *stopping policy* defined below.

Definition 3.1. *A function $\tau : \mathbb{X} \mapsto \mathcal{T}$ is called a stopping policy if $\tau(t,x) \in \mathcal{T}_t$ for all $(t,x) \in \mathbb{X}$, and*

$$\ker(\tau) := \{(t,x,\omega) \in \mathbb{X} \times \Omega : \tau(t,x)(\omega) = t\} \in \mathcal{B}(\mathbb{X}) \times \mathcal{F}_\infty. \tag{3.1}$$

We call $\ker(\tau)$ *the kernel of* τ*, and denote by* $\mathcal{T}(\mathbb{X})$ *the set of all stopping policies.*

The idea behind is as follows: given initial $(t,x)\in\mathbb{X}$, an agent with a policy $\tau\in\mathcal{T}(\mathbb{X})$ will modify her choice of stopping times at each moment $s\geq t$ according to τ. At $s\geq t$, she employs the stopping time $\tau(s,X_s(\omega))\in\mathcal{T}_s$. If $\tau(s,X_s(\omega))=s$, she stops right away; otherwise she continues. It follows that the moment eventually she will stop, when employing $\tau\in\mathcal{T}(\mathbb{X})$, is

$$\begin{aligned}\mathcal{L}\tau(t,x)(\omega) &:= \inf\left\{s\geq t\ :\ \tau(s,X_s^{t,x}(\omega))(\omega)=s\right\}\\ &= \inf\left\{s\geq t\ :\ (s,X_s^{t,x}(\omega),\omega)\in\ker(\tau)\right\}.\end{aligned}\tag{3.2}$$

With (3.1), the map $(t,x,\omega)\mapsto\mathcal{L}\tau(t,x)(\omega)$ from $(\mathbb{X}\times\Omega,\mathcal{B}(\mathbb{X})\times\mathcal{F}_\infty)$ to $(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$ is measurable. This particularly ensures that $\mathcal{L}\tau(t,x)$ is a well-defined stopping time.

Lemma 3.1. $\mathcal{L}\tau(t,x)\in\mathcal{T}_t$ *for all* $(t,x)\in\mathbb{X}$.

Proof. For any $s\geq t$, $E:=\{(t,x,\omega)\in\mathbb{X}\times\Omega:\mathcal{L}\tau(t,x)(\omega)<s\}\in\mathcal{B}(\mathbb{X})\times\mathcal{F}_\infty$. It follows that, for any fixed $(t,x)\in\mathbb{X}$, the (t,x)-section $E_{(t,x)}:=\{\omega\in\Omega:\mathcal{L}\tau(t,x)(\omega)<s\}\in\mathcal{F}_\infty$. Note that by definition of $\mathcal{L}\tau(t,x)$, $\{\omega\in\Omega:\mathcal{L}\tau(t,x)(\omega)<s\}$ is independent of $\sigma(W_u:u\leq\ell)$ for all $\ell<t$ and $\sigma(W_u:u\geq r)$ for all $r>s$. Thus, $\{\omega\in\Omega:\mathcal{L}\tau(t,x)(\omega)<s\}\in\mathcal{F}_s^t$. ☐

Remark 3.1 (Naive Stopping Policy)**.** *The function* $\widetilde{\tau}:\mathbb{X}\mapsto\mathcal{T}$*, defined in* (2.8)*, belongs to* $\mathcal{T}(\mathbb{X})$*. Indeed,* $\widetilde{\tau}(t,x)=t$ *if and only if*

$$(t,x)\in A:=\left\{(t,x)\in\mathbb{X}:g(x)=\sup_{\tau\in\mathcal{T}_t}\mathbb{E}[\delta(\tau-t)g(X_\tau^{t,x})]\right\}\in\mathcal{B}(\mathbb{X}).$$

It follows that $\ker(\widetilde{\tau})=A\times\Omega\in\mathcal{B}(\mathbb{X})\times\mathcal{F}_\infty$*. We call* $\widetilde{\tau}$ *the naive stopping policy, as it is the policy followed by a naive agent, as discussed in Section 2.3.*

Example 3.1 (Real Option Model, Continued)**.** *Recall the setting of Example 2.2. Considering* $\widetilde{\tau}\in\mathcal{T}(\mathbb{X})$ *given by* (2.16)*, we note that the actual stopping time in effect is*

$$\mathcal{L}\tau(t,x)=\inf\{s\geq t:\widetilde{\tau}(s,X_s^{t,x})=s\}=\inf\{s\geq t:|X_s^{t,x}|\geq 1\},$$

which is different from the original decision $\widetilde{\tau}(t,x)$ *in* (2.16)*.*

We can now lay the ground for equilibrium policies. Suppose that an agent is given a stopping policy $\tau\in\mathcal{T}(\mathbb{X})$. At each $(t,x)\in\mathbb{X}$, we assume that the agent considers the problem: "assuming that all my future selves will follow the policy τ, how can I improve my stopping time $\tau(t,x)$ today at time t?" Notice that the agent at time t has only two possible actions: stopping and continuation. If she stops at time t, she gets $g(x)$ immediately. If she continues at time t, since her future selves are assumed to employ $\tau(s,X_s^{t,x})\in\mathcal{T}_s$ for all $s>t$, she will eventually stop at the moment

$$\begin{aligned}\mathcal{L}^*\tau(t,x)(\omega) &:= \inf\left\{s>t\ :\ \tau(s,X_s^{t,x}(\omega))(\omega)=s\right\}\\ &= \inf\left\{s>t\ :\ (s,X_s^{t,x}(\omega),\omega)\in\ker(\tau)\right\},\end{aligned}\tag{3.3}$$

and her expected payoff is

$$J(t,x;\mathcal{L}^*\tau(t,x))=\mathbb{E}^{t,x}\left[\delta(\mathcal{L}^*\tau(t,x)-t)g(X_{\mathcal{L}^*\tau(t,x)})\right].$$

Intuitively, if the agent at time t chooses to continue, she does not care about what $\tau\in\mathcal{T}(\mathbb{X})$ tells her to do at time t. She starts to follow τ only when she goes beyond time t. This is why we have "$s>t$" in (3.3), instead of "$s\geq t$" as in (3.2). As in Lemma 3.1, under (3.1), the map $(t,x,\omega)\mapsto\mathcal{L}^*\tau(t,x)(\omega)$ from $(\mathbb{X}\times\Omega,\mathcal{B}(\mathbb{X})\times\mathcal{F}_\infty)$ to $(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$ is measurable, which

guarantees that $\mathcal{L}^*\tau(t,x) \in \mathcal{T}_t$ for all $(t,x) \in \mathbb{X}$. As a result, $J(t,x;\mathcal{L}^*\tau(t,x))$ in the above equation is well-defined. Some simple properties of $\mathcal{L}$ and $\mathcal{L}^*$ can be found in Lemma A.1.

Now, we separate the space $\mathbb{X}$ into three distinct regions

$$
\begin{aligned}
S_\tau &:= \{(t,x) \in \mathbb{X} \ : \ g(x) > J(t,x;\mathcal{L}^*\tau(t,x))\},\\
C_\tau &:= \{(t,x) \in \mathbb{X} \ : \ g(x) < J(t,x;\mathcal{L}^*\tau(t,x))\},\\
I_\tau &:= \{(t,x) \in \mathbb{X} \ : \ g(x) = J(t,x;\mathcal{L}^*\tau(t,x))\}.
\end{aligned}
\tag{3.4}
$$

For any $(t,x) \in \mathbb{X}$, assuming that all future selves will follow $\tau \in \mathcal{T}(\mathbb{X})$, we can improve the stopping time $\tau(t,x)$ at time t as follows:

1. If $(t,x) \in S_\tau$, the agent stops immediately at time t.
2. If $(t,x) \in C_\tau$, the agent continues at time t, and expects to stop at $\mathcal{L}^*\tau(t,x)$.
3. If $(t,x) \in I_\tau$, the agent keeps her strategy $\tau(t,x)$, and expects to stop at $\mathcal{L}\tau(t,x)$.

To summarize, for any $\tau \in \mathcal{T}(\mathbb{X})$ and $(t,x) \in \mathbb{X}$, we introduce

$$
\Theta\tau(t,x) := \begin{cases} t & \text{for } (t,x) \in S_\tau \\ \mathcal{L}\tau(t,x) & \text{for } (t,x) \in I_\tau \\ \mathcal{L}^*\tau(t,x) & \text{for } (t,x) \in C_\tau \end{cases}.
\tag{3.5}
$$

Lemma 3.2. *For any $\tau \in \mathcal{T}(\mathbb{X})$, the regions S_τ, C_τ, and I_τ belong to $\mathcal{B}(\mathbb{X})$, and the map $(t,x,\omega) \mapsto \Theta\tau(t,x)(\omega)$ from $(\mathbb{X}\times\Omega, \mathcal{B}(\mathbb{X})\times\mathcal{F}_\infty)$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ is measurable. In particular, $\Theta\tau \in \mathcal{T}(\mathbb{X})$.*

Proof. From Lemma 3.1 and the discussion under (3.3), we know (i) $(t,x,\omega) \mapsto \mathcal{L}\tau(t,x)(\omega)$ is measurable; (ii) $(t,x,\omega) \mapsto \mathcal{L}^*\tau(t,x)(\omega)$ is measurable; (iii) $\mathcal{L}\tau(t,x) \in \mathcal{T}_t$ and $\mathcal{L}^*(t,x) \in \mathcal{T}_t$ for all $(t,x) \in \mathbb{X}$. By Fubini's theorem and (ii), the map $(t,x) \mapsto J(t,x;\mathcal{L}^*\tau(t,x)) = \mathbb{E}[\delta(\mathcal{L}^*\tau(t,x) - t)g(X^{t,x}_{\mathcal{L}^*\tau(t,x)})]$ is measurable. It follows that S_τ, I_τ, and C_τ all belong to $\mathcal{B}(\mathbb{X})$. Now, we observe from (3.5) that (i), (ii), and the Borel measurability of S_τ, I_τ, and C_τ imply that $(t,x,\omega) \mapsto \Theta\tau(t,x)(\omega)$ is measurable. This in particular yields $\ker(\Theta\tau) = \{(t,x,\omega) \in \mathbb{X}\times\Omega : \Theta\tau(t,x)(\omega) = t\} \in \mathcal{B}(\mathbb{X})\times\mathcal{F}_\infty$. Since (iii) immediately gives $\Theta\tau(t,x) \in \mathcal{T}_t$ for all $(t,x) \in \mathbb{X}$, we conclude that $\Theta\tau \in \mathcal{T}(\mathbb{X})$. ☐

By Lemma 3.2, $\Theta : \mathcal{T}(\mathbb{X}) \mapsto \mathcal{T}(\mathbb{X})$ generates a new policy $\Theta\tau \in \mathcal{T}(\mathbb{X})$ from any initial $\tau \in \mathcal{T}(\mathbb{X})$. Notice that we obtain $\Theta\tau$ from a one-level strategic reasoning, i.e. the local improvements of τ at all $(t,x) \in \mathbb{X}$. The following definition thus follows naturally.

Definition 3.2 (Equilibrium Stopping Policies)**.** *We say $\tau \in \mathcal{T}(\mathbb{X})$ is an equilibrium stopping policy if $\Theta\tau(t,x) = \tau(t,x)$ a.s. for all $(t,x) \in \mathbb{X}$. We denote by $\mathcal{E}(\mathbb{X})$ the collection of all equilibrium stopping policies.*

The term "equilibrium" is used in Definition 3.2 as a connection to subgame perfect Nash equilibria that are also invoked in the stochastic control literature under time inconsistency; see Ekeland and Lazrak (2006), Ekeland and Pirvu (2008), Ekeland et al. (2012), and Björk and Murgoci (2014b), among others. In the present case, the inter-temporal game is described among players characterized by each $(t,x) \in \mathbb{X}$. The sub-game Markov property is inherent in Definition 3.1, in echo to the setup of $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq t}$ in Section 2.1. The Nash equilibrium property relates to the fixed point property in Definition 3.2. Nevertheless, those relations are purely formal, and we simply stick to our definitions. The Nash equilibrium property brings the next remark.

Remark 3.2. *When $(t,x) \in I_\tau$, the agent is indifferent between stopping right away and the status quo $\tau(t,x)$. The former choice could be justified if one affirms that "the sooner she stops the better". The later is justified by the absence of incentive to deviate from the assigned $\tau(t,x)$, which is at the root of what defines a Nash equilibrium. From a technical point of view, the later choice is also less restrictive for obtaining fixed points of Θ.*

A contrast with the stochastic control literature needs to be pointed out.

Remark 3.3 (Comparison to Stochastic Control)**.** *The way we formulate equilibrium in Definition 3.2 differs from previous research on time inconsistency. In stochastic control, local perturbation of strategies on small time intervals $[t, t+\varepsilon]$ is a standard technique. As $\varepsilon \to 0$, it provides a characterization of the equilibrium solution, or IPDEs; see Ekeland et al. (2013). In our case, local perturbation is carried out instantaneously at time t. This is because an instantaneously-modified stopping strategy may already change the expected discounted payoff significantly (whereas a control perturbed only at time t generally yields no effect).*

After providing Definition 3.2, we must ask ourselves about the existence of an equilibrium stopping policies. Finding at least one element in $\mathcal{E}(\mathbb{X})$ turns out to be easy.

Remark 3.4 (Trivial Equilibrium)**.** *Define $\tau \in \mathcal{T}(X)$ by $\tau(t,x) := t$ for all $(t,x) \in \mathbb{X}$. Then, it can be checked that $\mathcal{L}\tau(t,x) = \mathcal{L}^*\tau(t,x) = t$ for all $(t,x) \in \mathbb{X}$. In view of (3.5), we conclude that $\Theta\tau(t,x) = t = \tau(t,x)$ for all $(t,x) \in \mathbb{X}$, which shows $\tau \in \mathcal{E}(\mathbb{X})$. We call it the trivial equilibrium stopping policy.*

Example 3.2 (Smoking Cessation, Continued)**.** *Recall the setting in Example 2.1. Observe from (2.15) that $\mathcal{L}\widetilde{\tau}(t,x) = \mathcal{L}^*\widetilde{\tau}(t,x) = T$. Then,*

$$\delta(\mathcal{L}^*\widetilde{\tau}(t,x) - t) X^{t,x}_{\mathcal{L}^*\widetilde{\tau}(t,x)} = \frac{X^{t,x}_T}{1+T-t} = \frac{x e^{\frac{1}{2}(T-t)}}{1+T-t}.$$

Since $e^{\frac{1}{2}s} = 1+s$ at $s = 0, s^*$, with $s^* \approx 2.51286$, and $e^{\frac{1}{2}s} > 1+s$ iff $s > s^*$, the above equation implies $S_{\widetilde{\tau}} = \{(t,x) : t < T - s^*\}$, $C_{\widetilde{\tau}} = \{(t,x) : t \in (T-s^*, T)\}$, and $I_{\widetilde{\tau}} = \{(t,x) : t = T - s^* \text{ or } T\}$. We therefore get

$$\Theta\widetilde{\tau}(t,x) = \begin{cases} t & \text{for } t < T - s^*, \\ T & \text{for } t \geq T - s^*. \end{cases}$$

To conclude, in this model, whereas a naive smoker will procrastinate the quitting decision, a first degree of rationality (i.e. applying Θ to $\widetilde{\tau}$ once) recognizes this procrastination behavior and pushes the smoker to quit immediately, unless he is already very close to death time T (i.e. $t \geq T - s^*$). It can be checked that this decision $\Theta\widetilde{\tau}$ is an equilibrium, i.e. $\Theta^2\widetilde{\tau}(t,x) = \Theta\widetilde{\tau}(t,x)$ for all (t,x).

It is worth noting that under the classical case of exponential discounting, characterized by the identity (2.12), the connection between the naive and sophisticated agents becomes trivial: the naive stopping policy $\widetilde{\tau} \in \mathcal{T}(\mathbb{X})$ is already an equilibrium.

Proposition 3.1. *Under (2.12), $\widetilde{\tau} \in \mathcal{T}(\mathbb{X})$, defined in (2.8), belongs to $\mathcal{E}(\mathbb{X})$.*

Proof. The proof is delegated to Appendix A.1. ∎

3.2 The Main Result

In this section, we look for equilibrium policies through fixed-point iterations. For any $\tau \in \mathcal{T}(\mathbb{X})$, we apply Θ on τ repetitively until we reach an equilibrium policy. In short, we define τ_0 by

$$\tau_0(t,x) := \lim_{n\to\infty} \Theta^n \tau(t,x) \quad \forall (t,x) \in \mathbb{X}, \tag{3.6}$$

and take it as a candidate equilibrium policy. To make this argument precise, we need to show (i) the limit in (3.6) converges, so that τ_0 is well-defined; (ii) τ_0 is indeed an equilibrium policy, i.e. $\Theta\tau_0 = \tau_0$. To ensure that the limit in (3.6) converges, we impose two conditions.

Assumption 3.1. *The function δ satisfies $\delta(s)\delta(t) \leq \delta(s+t)$ for all $s,t \geq 0$.*

Let us first elucidate the relation between Assumption 3.1 and *decreasing impatience*. In Behavioral Economics and Finance, it is well-documented from empirical evidence that people admits decreasing impatience (DI). See e.g. Thaler (1981), Loewenstein and Thaler (1989), and Loewenstein and Prelec (1992). When choosing between two rewards, people are more willing to wait for the larger reward (more patient) when these two rewards are further away in time. For instance, in the two scenarios (i) getting \$100 today or \$110 tomorrow, and (ii) getting \$100 in 100 days or \$110 in 101 days, people tend to choose \$100 in (i), but \$110 in (ii). Following Prelec (2004, Definition 1) and Noor (2009a,b), we define DI under current context as follows: the discount function δ induces DI if

$$\text{for any } s \geq 0, \quad t \mapsto \frac{\delta(t+s)}{\delta(t)} \quad \text{is strictly increasing.} \tag{3.7}$$

Observe that (3.7) readily implies Assumption 3.1, since $\delta(t+s)/\delta(t) \geq \delta(s)/\delta(0) = \delta(s)$ for all $s,t \geq 0$. That is, Assumption 3.1 is automatically true under DI. Note that Assumption 3.1 is more general than DI, as it obviously includes the case of exponential discounting. As a result, all that is stated here in this section holds for the usual exponential case.

Assumption 3.2. *Let $\tau \in \mathcal{T}(\mathbb{X})$ satisfy*

$$\mathcal{L}^*\Theta\tau(t,x) \leq \mathcal{L}^*\tau(t,x) \quad a.s. \quad \forall (t,x) \in \mathbb{X}. \tag{3.8}$$

It is noticeable that the naive stopping policy $\widetilde{\tau}$ satisfies the above technical assumption.

Proposition 3.2. *$\widetilde{\tau} \in \mathcal{T}(\mathbb{X})$ defined in (2.8) satisfies Assumption 3.2.*

Proof. The proof is delegated to Appendix A.2. ∎

Assumptions 3.1 and 3.2 together imply the monotonicity of the iteration $\{\Theta^n\tau\}_{n\in\mathbb{N}}$.

Proposition 3.3. *Let Assumption 3.1 hold. If $\tau \in \mathcal{T}(\mathbb{X})$ satisfies Assumption 3.2, then*

$$\Theta^{n+1}\tau(t,x)(\omega) \leq \Theta^n\tau(t,x)(\omega) \quad \forall (t,x,\omega) \in \mathbb{X}\times\Omega \text{ and } n \in \mathbb{N}. \tag{3.9}$$

Hence, τ_0 in (3.6) is a well-defined element in $\mathcal{T}(\mathbb{X})$.

Proof. The proof is delegated to Appendix A.3. ∎

The next theorem is the main result of our paper. It shows that the iteration $\{\Theta^n\tau\}_{n\in\mathbb{N}}$ indeed converges to an equilibrium policy.

Theorem 3.1. *Let Assumption 3.1 hold. If $\tau \in \mathcal{T}(\mathbb{X})$ satisfies Assumption 3.2, then τ_0 defined in (3.6) belongs to $\mathcal{E}(\mathbb{X})$.*

Proof. The proof is delegated to Section A.4. ∎

The following result of the naive stopping policy $\widetilde{\tau}$, defined in (2.8), is a direct consequence of Proposition 3.2 and Theorem 3.1.

Corollary 3.1. *Let Assumption 3.1 hold. The stopping policy $\widetilde{\tau}_0 \in \mathcal{T}(\mathbb{X})$ defined by*

$$\widetilde{\tau}_0(t,x) := \lim_{n\to\infty} \Theta^n \widetilde{\tau}(t,x) \quad \forall (t,x) \in \mathbb{X} \tag{3.10}$$

belongs to $\mathcal{E}(\mathbb{X})$.

As stated in Introduction, Corollary 3.1 is a significant result for the literature of time inconsistency. First, it states an explicit relation between the (irrational) naive behavior and the (perfectly rational) sophisticated one. Moreover, the "improving" procedure in (3.10) corresponds to the increasing levels of strategic thinking of an agent, in anticipation of her future behavior. A naive agent follows classical optimal stopping times without any regard to her future behavior, whereas a sophisticated one, foreseeing all the pattern of her future behavior, seeks a Nash equilibrium. Between them, there are many degrees of rationality presented here: for each $n \in \mathbb{N}$, $\Theta^n \tau$ can be viewed as a stopping behavior under bounded (or limited) rationality, a notion introduced in Simon (1982).

4 A Detailed Case Study: Stopping of BES(1)

In this section, we recall the setup of Example 2.2, in the context of real irreversible investment (entry or exit) with hyperbolic time preferences:

$$\delta(s) := \frac{1}{1+\beta s} \quad \forall s \geq 0, \tag{4.1}$$

where $\beta > 0$ is a fixed parameter. The price process $\{X_t\}_{t\geq 0}$ is a one-dimensional Brownian motion and $g(x) := |x|$ is the payoff function. Equivalently, $\{|X_t|\}_{t\geq 0}$ can be used model the price process, which is a one-dimensional Bessel process. Here, we aim to characterize the naive policy, the set of equilibrium policies, and possibly all intermediary policies obtained through repetitive application of Θ. We comment on the behavioral implications.

Given initial time and state $(t,x) \in \mathbb{X} := \mathbb{R}_+ \times \mathbb{R}$, the classical optimal stopping problem (2.6) becomes

$$v(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}^{t,x}\left[\frac{|X_\tau|}{1+\beta(\tau - t)}\right]. \tag{4.2}$$

An unusual feature of (4.2) is that the discounted process $\{\delta(s-t)v(s, X_s^{t,x})\}_{s\geq t}$ may not be a supermartingale. This makes solving (4.2) for the optimal stopping time $\widetilde{\tau}(t,x)$, defined in (2.8), a nontrivial task. As demonstrated in Appendix B.1, we need to introduce auxiliary value functions, and employ the method of time-change as in Pederson and Perkir (2000). The result is the following.

Proposition 4.1. *For any $(t,x) \in \mathbb{X}$, the optimal stopping time $\widetilde{\tau}(t,x)$ of (4.2) (defined in (2.8)) admits the explicit formula*

$$\widetilde{\tau}(t,x) = \inf\left\{ s \geq t : |X_s^{t,x}| \geq \sqrt{1/\beta + (s-t)} \right\}. \tag{4.3}$$

Proof. The proof is delegated to Appendix B.1. □

4.1 Characterization of equilibrium policies

Given a stopping policy $\tau \in \mathcal{T}(\mathbb{X})$, we know from Section 3 that $\tau(t,x)$ indicates stopping or continuation at each $(t,x) \in \mathbb{X}$. Under current setting, since X is a time-homogeneous process, it is reasonable to restrict ourselves to stopping policies τ that are *time-invariant*, i.e. $\tau(t,x) = \tau(x)$ for all $(t,x) \in \mathbb{X}$. In addition, in view of (4.2), there is no distinction between a given path $X^{t,x}(\omega)$ and its opposite $X^{t,-x}(-\omega)$. We are therefore led to focus on stopping policies $\tau \in \mathcal{T}(\mathbb{X})$ of the form

$$\tau(t,x) = \inf\left\{s \geq t \ : \ |X_s^{t,x}| \in L\right\} \quad \forall (t,x) \in X, \quad \text{for some } L \in \mathcal{B}(\mathbb{R}_+), \tag{4.4}$$

where $\mathcal{B}(\mathbb{R}_+)$ is the Borel σ-algebra of $\mathbb{R}_+$. We denote by $\mathcal{T}'(\mathbb{X})$ the collection of all $\tau \in \mathcal{T}(\mathbb{X})$ of the form (4.4), and consider the collection

$$\mathcal{E}'(\mathbb{X}) := \{\tau \in \mathcal{T}'(\mathbb{X}) \ : \ \Theta\tau(t,x) = \tau(t,x) \quad a.s. \quad \forall (t,x) \in \mathbb{X}\} = \mathcal{T}'(\mathbb{X}) \cap \mathcal{E}(\mathbb{X}).$$

Remark 4.1. *When the SDE (2.3) is time-homogeneous, whether $\mathcal{E}(\mathbb{X})$ and $\mathcal{E}'(\mathbb{X})$ actually coincide is an interesting question of its own. We leave this for future research.*

Lemma 4.1. *For any $\tau \in \mathcal{E}'(\mathbb{X})$, there exists $a \geq 0$ such that*

$$\tau(t,x) = \tau_a(t,x) := \inf\left\{s \geq t \ : \ |X_s^{t,x}| \geq a\right\} \quad (t,x) \in \mathbb{X}. \tag{4.5}$$

Proof. The proof is delegated to Appendix B.2 $\square$

The next question is for which $a \geq 0$ can the policy τ_a be an equilibrium. To answer this, we need to determine the three sets S_{τ_a}, C_{τ_a}, and I_{τ_a} in (3.4). First, by (3.2) and (3.3),

$$\begin{aligned} \mathcal{L}\tau_a(t,x) &= \inf\{s \geq t : |X_s^{t,x}| \geq a\} = \tau_a(t,x), \\ \mathcal{L}^*\tau_a(t,x) &= \inf\{s > t : |X_s^{t,x}| \geq a\}. \end{aligned} \tag{4.6}$$

Observe that

$$\mathcal{L}^*\tau_a(t,x) = t \quad \text{if } |x| > a. \tag{4.7}$$

Thus, when $|x| > a$, we have $J(t,x;\mathcal{L}^*\tau_a(t,x)) = |x|$, and it follows that

$$\{(t,x) \in \mathbb{X} \ : \ |x| > a\} \subseteq I_{\tau_a}. \tag{4.8}$$

If $|x| = a$, then $\mathbb{P}[\mathcal{L}^*\tau_a(t,x) > t] > 0$ and thus

$$J(t,a;\mathcal{L}^*\tau_a(t,x)) = \mathbb{E}_{t,x}\left[a\mathbf{1}_{\{\mathcal{L}^*\tau_a(t,x)=t\}} + \frac{a\mathbf{1}_{\{\mathcal{L}^*\tau_a(t,x)>t\}}}{1+\beta(\mathcal{L}^*\tau_a(t,x)-t)}\right] < a = |x|. \tag{4.9}$$

This implies

$$\{(t,x) \in \mathbb{X} \ : \ |x| = a\} \subseteq S_{\tau_a}. \tag{4.10}$$

If $|x| < a$, we need to use the lemma below, whose proof is delegated to Appendix B.3.

Lemma 4.2. *On the space $\{(x,a) \in \mathbb{R}_+^2 \ : \ a \geq x\}$, define the function*

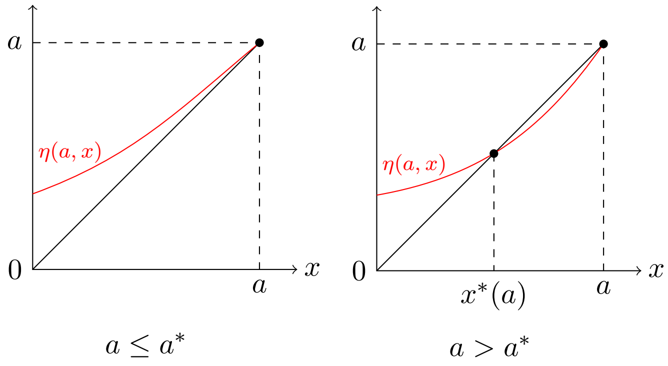
$$\eta(x,a) := \mathbb{E}_{t,x}\left[\frac{a}{1+\beta(\tau_a(t,x)-t)}\right].$$

(i) For any $a \geq 0$, $x \mapsto \eta(x,a)$ is strictly increasing and strictly convex on [0,a], and satisfies $0 < \eta(0,a) < a$ and $\eta(a,a) = a$.

(ii) For any $x \geq 0$, $\eta(x, a) \to 0$ as $a \to \infty$.

(iii) There exists a unique $a^* \in (0, 1/\sqrt{\beta})$ such that for any $a > a^*$, there is a unique solution $x^*(a) \in (0, a^*)$ of $\eta(x, a) = x$. Hence, $\eta(x, a) > x$ for $x < x^*(a)$ and $\eta(x, a) < x$ for $x > x^*(a)$. On the other hand, $a \leq a^*$ implies that $\eta(x, a) > x$ for all $x \in (0, a)$.

The figure below illustrates $x \mapsto \eta(a, x)$ under different scenarios $a \leq a^*$ and $a > a^*$.



We now separate the case $|x| < a$ into two sub-cases:

1. If $a \leq a^*$, then Lemma 4.2 (iii) shows that $J(t, a; \mathcal{L}^*\tau_a(t, x)) = \eta(a, x) > |x|$, and thus

$$\{(t, x \in \mathbb{X}) \ : \ |x| < a\} \subseteq C_{\tau_a}. \tag{4.11}$$

2. If $a > a^*$, then by Lemma 4.2 (iii),

$$J(t, a; \mathcal{L}^*\tau_a(t, x)) = \eta(a, x) \begin{cases} > |x|, & \text{if } x \leq x^*(a), \\ = |x|, & \text{if } x = x^*(a), \\ < |x|, & \text{if } x \in (x^*(a), a). \end{cases} \tag{4.12}$$

With (4.6)-(4.11), the definition of Θ in (3.5) implies

$$\Theta\tau_a(t, x) = \inf\{s \geq t \ : \ |X_s^{t,x}| \geq a\} = \tau_a(t, x), \quad \text{if } a \leq a^*.$$

Similarly, with (4.6)-(4.10) and (4.12), the definition of Θ in (3.5) implies

$$\Theta\tau_a(t, x) = t\mathbf{1}_{\{|x| > x^*(a)\}} + \tau_a(t, x)\mathbf{1}_{\{|x| \leq x^*(a)\}} \neq \tau_a(t, x), \quad \text{if } a > a^*. \tag{4.13}$$

This shows that we have characterized the entire collection $\mathcal{E}'(\mathbb{X})$.

Proposition 4.2. *$\mathcal{E}'(\mathbb{X}) = \{\tau_a \in \mathcal{T}(\mathbb{X}) \ : \ a \in [0, a^*]\}$, for $a^* > 0$ being the solution to*

$$a \int_0^\infty e^{-s} \sqrt{2\beta s} \tanh(a\sqrt{2\beta s}) ds = 1. \tag{4.14}$$

Proof. The existence of a^* and the characterization of $\mathcal{E}'(\mathbb{X})$ is presented in the discussion above this proposition. To find a^*, by the proof of Lemma 4.2 in Appendix B.3, one needs to solve $\eta_a(a, a) = 1$ for a, which leads to (4.14). $\square$

Remark 4.2 (Estimating a^*)**.** *With $\beta = 1$, numerical computation of (4.14) gives $a^* \approx 0.946475$. It follows that for a general $\beta > 0$, $a^* \approx 0.946475/\sqrt{\beta}$.*

4.2 Construction of equilibrium policies

When $a > a^*$, although $\tau_a \notin \mathcal{E}'(\mathbb{X})$ by Proposition 4.2, we may employ the iteration in Theorem 3.1 to find a stopping policy in $\mathcal{E}'(\mathbb{X})$. In general, the repetitive application of Θ in Theorem 3.1 can be very complicated. It has, nonetheless, a simple structure under current setting: to reach an equilibrium, we need at most *two* iterations. To establish this, we need the next lemma, whose proof is delegated in Appendix B.4.

Lemma 4.3. *For any $0 \le x < a$ and $t \ge 0$, $\tau_a(t,x) = \bar{\tau}_a(t,x) := \inf\{s \ge t : |X_s^{t,x}| > a\}$ a.s.*

Proposition 4.3. *Let $a > a^*$. Then*

$$\widehat{\tau}_a := \lim_{n\to\infty} \Theta^n \tau_a = \Theta^2 \tau_a = \tau_{x^*(a)} \in \mathcal{E}'(\mathbb{X}), \tag{4.15}$$

where $x^*(a) \in (0, a^*)$ is given in Lemma 4.2 (iii)

Proof. Note that $x^*(a) < a^* < a$. We deduce from (4.13) that

$$\mathcal{L}\Theta\tau_a(t,x) = \inf\{s \ge t : |X_s^{t,x}| > x^*(a)\}, \quad \mathcal{L}^*\Theta\tau_a(t,x) = \inf\{s > t : |X_s^{t,x}| > x^*(a)\}. \tag{4.16}$$

It follows that

$$\mathcal{L}^*\tau_a(t,x) = \inf\{s > t : |X_s^{t,x}| \ge a\} \ge \inf\{s \ge t : |X_s^{t,x}| > x^*(a)\} = \mathcal{L}^*\Theta\tau_a(t,x).$$

That is, τ_a satisfies (3.8). We then conclude from Theorem 3.1 that $\widehat{\tau}_a \in \mathcal{T}(\mathbb{X})$ defined by

$$\widehat{\tau}_a(t,x) := \lim_{n\to\infty} \Theta^n \tau_a(t,x) \quad \forall (t,x) \in \mathbb{X}$$

belongs to $\mathcal{E}(\mathbb{X})$. The first iteration is already given in (4.13). From this, we deduce that $\{(t,x) \in \mathbb{X} : |x| > x^*(a)\} \subseteq I_{\Theta\tau_a}$ and $\{(t,x) \in \mathbb{X} : |x| = x^*(a)\} \subseteq S_{\Theta\tau_a}$, just as how we derive (4.8) and (4.10). To determine $C_{\Theta\tau_a}$, Lemma 4.3 and Lemma 4.2 imply that, if $|x| < x^*(a)$,

$$J(t,x;\mathcal{L}^*\Theta\tau_a(t,x)) = J(t,x;\bar{\tau}_{x^*(a)}(t,x)) = J(t,x;\tau_{x^*(a)}(t,x)) > |x|.$$

Thus, $\{(t,x) \in \mathbb{X} : |x| < x^*(a)\} \subseteq C_{\Theta\tau_a}$. We then conclude from (4.16) that

$$\Theta^2\tau_a(t,x) = t\mathbf{1}_{\{|x| \ge x^*(a)\}} + \bar{\tau}_{x^*(a)}\mathbf{1}_{\{|x| < x^*(a)\}} = \tau_{x^*(a)}(t,x) \quad \text{a.s.},$$

where the second equality follows again from Lemma 4.3. In view of Proposition 4.2, $\tau_{x^*(a)}$ is already an equilibrium policy in $\mathcal{E}'(\mathbb{X})$. $\square$

As emphasized under Corollary 3.1, (3.10) connects the naive and sophisticated agents. Under current setting, with the naive strategy $\widetilde{\tau} \in \mathcal{T}(\mathbb{X})$ given explicitly in (4.3), we intend to find a formula for $\widetilde{\tau}_0 := \lim_{n\to\infty} \Theta^n \widetilde{\tau}$ as explicit as possible. Set $\widetilde{a} := 1/\sqrt{\beta}$. We observe from (4.3) and (4.6) that

$$\begin{aligned} \mathcal{L}\widetilde{\tau}(t,x) &= \inf\{s \ge t : |X_s^{t,x}| \ge \widetilde{a}\} = \tau_{\widetilde{a}}(t,x) = \mathcal{L}\tau_{\widetilde{a}}(t,x), \\ \mathcal{L}^*\widetilde{\tau}(t,x) &= \inf\{s > t : |X_s^{t,x}| \ge \widetilde{a}\} = \mathcal{L}^*\tau_{\widetilde{a}}(t,x). \end{aligned}$$

It follows that $\Theta\widetilde{\tau}(t,x) = \Theta\tau_{\widetilde{a}}(t,x)$. Note that $\widetilde{a} > a^*$ from Remark 4.2. We may then apply Proposition 4.3 to $\tau_{\widetilde{a}}$. This gives $\widetilde{\tau}_0(t,x) = \lim_{n\to\infty} \Theta^n \tau_{\widetilde{a}}(t,x) = \Theta^2\tau_{\widetilde{a}}(t,x) = \tau_{x^*(\widetilde{a})}(t,x)$. By the proof of Lemma 4.2 in Appendix B.3, in order to find $x^*(\widetilde{a})$, we need to solve the equation

$$\eta(1/\sqrt{\beta}, x) = \frac{1}{\sqrt{\beta}} \int_0^\infty e^{-s} \cosh(x\sqrt{2\beta s}) \operatorname{sech}(\sqrt{2s}) ds = x$$

for x. Numerical computation shows that $x^*(\widetilde{a}) = x^*(1/\sqrt{\beta}) \approx 0.92195/\sqrt{\beta}$. Since $x^*(1/\sqrt{\beta}) < a^*/\sqrt{\beta}$ for all $\beta > 0$, the equilibrium is reached in only one step.

Here, we draw a conclusion reminiscent of Grenadier and Wang (2007). Independently of the initial time, the payoff obtained at the moment of stopping is greater when using the naive policy (the payoff is $\widetilde{a}$) than using the equilibrium policy derived from it (the payoff is $x^*(\widetilde{a}) < \widetilde{a}$). We explain this behavior by the very explicit calculations in Proposition 4.3. When a reward is away from present, low impatience predominates in our computations. In a dynamic setting, an agent must anticipate her future eagerness to stop when the reward gets closer, and thus lower her expected payoff. From Proposition 4.1, we also notice that, at the moment of stopping, both of the above strategies provide lower payoffs than the pre-committed strategy, i.e. committing to the initial optimal stopping time.

4.3 Further considerations on selecting equilibrium policies

Given the characterization of $\mathcal{E}'(\mathbb{X})$ in Proposition 4.2, it is natural to ask which equilibrium policy in $\mathcal{E}'(\mathbb{X})$ one should employ. Given $(t,x) \in \mathbb{X}$, since a sophisticated agent intends to apply an equilibrium policy, the stopping problem she faces can be formulated as

$$\sup_{\tau \in \mathcal{E}'(\mathbb{X})} J(t,x;\tau(t,x)). \tag{4.17}$$

Under current setting, (4.17) becomes

$$\sup_{\tau \in \mathcal{E}'(\mathbb{X})} \mathbb{E}^{t,x}\left[\frac{|X_{\tau(t,x)}|}{1+\beta(\tau(t,x)-t)}\right] = \sup_{a \in [x, a^* \vee x]} \mathbb{E}^{t,x}\left[\frac{a}{1+\beta(\tau_a(t,x)-t)}\right], \tag{4.18}$$

where the second equality follows from Proposition 4.2.

Proposition 4.4. *$\tau_{a^*} \in \mathcal{E}(\mathbb{X})$ solves (4.18) for all $(t,x) \in \mathbb{X}$.*

Proof. Fix $a \in [0, a^*)$. For any $(t,x) \in \mathbb{X}$ with $x \le a \le a^*$, $\tau_a(t,x) \le \tau_{a^*}(t,x)$ a.s. We then have

$$\begin{aligned}
J(t,x;\tau_{a^*}(t,x)) &= \mathbb{E}^{t,x}\left[\frac{a^*}{1+\beta(\tau_{a^*}(t,x)-t)}\right] \ge \mathbb{E}^{t,x}\left[\frac{1}{1+\beta(\tau_a(t,x)-t)}\frac{a^*}{1+\beta(\tau_{a^*}(t,x)-t)}\right]\\
&= \mathbb{E}^{t,x}\left[\frac{1}{1+\beta(\tau_a(t,x)-t)}\mathbb{E}^{\tau_a(t,x),a}\left[\frac{a^*}{1+\beta(\tau_{a^*}(\tau_a(t,x),a)-\tau_a(t,x))}\right]\right]\\
&> \mathbb{E}^{t,x}\left[\frac{a}{1+\beta(\tau_a(t,x)-t)}\right] = J(t,x;\tau_a(t,x)),
\end{aligned}$$

where the last inequality follows from Lemma 4.2. $\square$

The conclusion is twofold. First, it is possible, at least under current setting, to find one single equilibrium policy that solves (4.18) *for all* $(t,x) \in \mathbb{X}$. Second, this "optimal" equilibrium policy τ_{a^*} is different from $\widetilde{\tau}_0 = \tau_{x^*(\widetilde{a})}$, obtained from the naive strategy $\widetilde{\tau}$ under strategic anticipation in (3.10). For a general objective function $J(t,x;\tau)$, there is thus a strong incentive for obtaining the whole set $\mathcal{E}'(\mathbb{X})$ (or $\mathcal{E}(\mathbb{X})$), instead of only applying Corollary 3.1.

A Proofs for Section 3

Throughout this Appendix, we will constantly use the notation

$$\tau_n := \Theta^n \tau \quad n \in \mathbb{N}, \quad \text{for any } \tau \in \mathcal{T}(\mathbb{X}). \tag{A.1}$$

Also, we collect some useful properties of $\mathcal{L}$ and $\mathcal{L}^*$ in the next lemma. We omit the proof as the properties are simple consequences of (3.2) and (3.3).

Lemma A.1. *We have*

a. $\tau(t,x) > t$ *implies* $\mathcal{L}\tau(t,x) = \mathcal{L}^*\tau(t,x)$;

b. *if* $\tau(t,x) = t$, *then* $t = \mathcal{L}\tau(t,x) \le \mathcal{L}^*\tau(t,x)$;

c. *if* $\mathcal{L}^*\tau(t,x)(\omega) = t$, *then there exist* $\{t_m\}_{m\in\mathbb{N}}$ *in* $\mathbb{R}_+$, *depending on* ω, *such that* $t_m \downarrow t$ *and* $\tau(t_m, X^{t,x}_{t_m}(\omega))(\omega) = t_m$ *for all* $m \in \mathbb{N}$.

d. *if* $\mathcal{L}^*\tau(t,x)(\omega) = s > t$, *then (a)* $\tau(s, X^{t,x}_s(\omega))(\omega) = s$ *or (b) there exist* $\{t_m\}_{m\in\mathbb{N}}$ *in* $\mathbb{R}_+$, *depending on* ω, *such that* $t_m \downarrow s$ *and* $\tau(t_m, X^{t,x}_{t_m}(\omega))(\omega) = t_m$ *for all* $m \in \mathbb{N}$.

A.1 Proof of Proposition 3.1

Fix $(t,x) \in \mathbb{X}$. If $\widetilde{\tau}(t,x) = t$, by (2.9) we have

$$g(x) = \sup_{\tau\in\mathcal{T}_t} \mathbb{E}^{t,x}[\delta(\tau - t)g(X_\tau)] \ge \mathbb{E}^{t,x}\left[\delta(\mathcal{L}^*\widetilde{\tau}(t,x) - t)g(X_{\mathcal{L}^*\widetilde{\tau}(t,x)})\right],$$

which implies $(t,x) \in S_{\widetilde{\tau}} \cup I_{\widetilde{\tau}}$. We then conclude from (3.5) that

$$\Theta\widetilde{\tau}(t,x) = \begin{cases} t & \text{if } (t,x) \in S_{\widetilde{\tau}} \\ \mathcal{L}\widetilde{\tau}(t,x) & \text{if } (t,x) \in I_{\widetilde{\tau}} \end{cases} = \widetilde{\tau}(t,x), \tag{A.2}$$

where $\mathcal{L}\widetilde{\tau}(t,x) = \widetilde{\tau}(t,x)$ follows from $\widetilde{\tau}(t,x) = t$.

Now, assume $\widetilde{\tau}(t,x) > t$. Under (2.12), with the same argument following it in Section 3.1, shows that $\widetilde{\tau}(t,x)(\omega) = \widetilde{\tau}(s, X^{t,x}_s(\omega))(\omega)$ for all $s \le \mathcal{L}\tau(t,x)(\omega)$, for a.e. $\omega \in \Omega$. Thus, $\mathcal{L}^*\widetilde{\tau}(t,x) = \mathcal{L}\widetilde{\tau}(t,x) = \inf\{s \ge t : \widetilde{\tau}(t,x) = s\} = \widetilde{\tau}(t,x)$ a.s. This, together with (2.9), shows that

$$\mathbb{E}^{t,x}\left[\delta(\mathcal{L}^*\widetilde{\tau}(t,x) - t)g(X_{\mathcal{L}^*\widetilde{\tau}(t,x)})\right] = \mathbb{E}^{t,x}\left[\delta(\widetilde{\tau}(t,x) - t)g(X_{\widetilde{\tau}(t,x)})\right] \ge g(x),$$

which implies $(t,x) \in I_{\widetilde{\tau}} \cup C_{\widetilde{\tau}}$. We then conclude from (3.5) that

$$\Theta\widetilde{\tau}(t,x) = \begin{cases} \mathcal{L}\widetilde{\tau}(t,x) & \text{if } (t,x) \in I_{\widetilde{\tau}} \\ \mathcal{L}^*\widetilde{\tau}(t,x) & \text{if } (t,x) \in C_{\widetilde{\tau}} \end{cases} = \widetilde{\tau}(t,x). \tag{A.3}$$

Combining the above two cases, we have $\Theta\widetilde{\tau}(t,x) = \widetilde{\tau}(t,x)$ a.s., as desired.

A.2 Proof of Proposition 3.2

Fix $(t,x) \in \mathbb{X}$. Set $t_0 := \mathcal{L}^*\widetilde{\tau}(t,x)$. For a.e. $\omega \in \Omega$, there are three possible situations at time $t_0(\omega)$. **Case (i):** $t_0(\omega) = t$. **Case (ii):** $t_0(\omega) > t$ and

$$\widetilde{\tau}(t_0(\omega), X^{t,x}_{t_0}(\omega))(\omega) = t_0(\omega). \tag{A.4}$$

Case (iii): $t_0(\omega) > t$, but (A.4) does not hold.

In the following, we first analyze Case (ii), and then deal with Cases (i) and (iii) together. **Case (ii):** By (2.8) and the right-continuity of $\{Z^{t,x}_s\}_{s\ge t}$, (A.4) implies

$$g(X^{t,x}_{t_0}(\omega)) = Z^{t_0(\omega), X^{t,x}_{t_0}(\omega)}_{t_0(\omega)}, \tag{A.5}$$

and thus

$$\mathcal{L}\widetilde{\tau}(t_0(\omega), X^{t,x}_{t_0}(\omega))(\omega) = t_0(\omega). \tag{A.6}$$

On the other hand, (A.5) and (2.7) in particular imply

$$g(X^{t,x}_{t_0}(\omega)) \ge \mathbb{E}^{t_0(\omega), X^{t,x}_{t_0}(\omega)}\left[\delta(\mathcal{L}^*\widetilde{\tau}(t_0, X^{t,x}_{t_0}) - t_0)g(X_{\mathcal{L}^*\widetilde{\tau}(t_0, X^{t,x}_{t_0})})\right],$$

which shows that $(t_0(\omega), X^{t,x}_{t_0}(\omega)) \notin C_{\widetilde{\tau}}$. We then conclude from (3.5) and (A.6) that

$$\text{(A.7)} \qquad \Theta\widetilde{\tau}(t_0(\omega), X^{t,x}_{t_0}(\omega)) = \begin{cases} t_0(\omega) & \text{if } (t_0(\omega), X^{t,x}_{t_0}(\omega)) \in S_{\widetilde{\tau}} \\ \mathcal{L}\widetilde{\tau}(t_0(\omega), X^{t,x}_{t_0}(\omega)) & \text{if } (t_0(\omega), X^{t,x}_{t_0}(\omega)) \in I_{\widetilde{\tau}} \end{cases} = t_0(\omega),$$

This, together with $t_0(\omega) > t$, shows that $\mathcal{L}^*\Theta\widetilde{\tau}(t,x)(\omega) \le t_0(\omega)$.
Cases (i) and (iii): In either case, there must exist $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$, depending on ω, such that $t_n \downarrow t_0(\omega)$ and $\widetilde{\tau}(t_n, X^{t,x}_{t_n}(\omega)) = t_n$ for all $n \in \mathbb{N}$. Thanks again to (2.8) and the right-continuity of $\{Z^{t,x}_s\}_{s\ge t}$, this implies

$$\text{(A.8)} \qquad g(X^{t,x}_{t_n}(\omega)) = Z^{t_n, X^{t,x}_{t_n}(\omega)}_{t_n} \quad \forall n \in \mathbb{N}.$$

With (A.8), we may argue as in Case (ii), with t_0 replaced by t_n, to show that $\Theta\widetilde{\tau}(t_n, X^{t,x}_{t_n}(\omega)) = t_n$ for all $n \in \mathbb{N}$. This implies $\mathcal{L}^*\Theta\widetilde{\tau}(t,x)(\omega) \le t_n$ for all $n \in \mathbb{N}$. As $n \to \infty$, we get $\mathcal{L}^*\Theta\widetilde{\tau}(t,x)(\omega) \le t_0(\omega)$.

A.3 Derivation of Proposition 3.3

We need the following technical result.

Lemma A.2. *For any $\tau \in \mathcal{T}(\mathbb{X})$ and $(t,x) \in \mathbb{X}$, define $t_0 := \mathcal{L}^*\tau_1(t,x) \in \mathcal{T}_t$ and $s_0 := \mathcal{L}^*\tau(t,x) \in \mathcal{T}_t$, with τ_1 as in (A.1). If $t_0 \le s_0$ a.s, then for a.e. $\omega \in \{t < t_0\}$,*

$$g(X^{t,x}_{t_0}(\omega)) \le \mathbb{E}\left[\delta(s_0 - t_0) g(X^{t,x}_{s_0}) \mid \mathcal{F}_{t_0}\right](\omega).$$

Proof. For a.e. $\omega \in \{t < t_0\} \in \mathcal{F}_t$, we deduce from $t_0(\omega) = \mathcal{L}^*\tau_1(t,x)(\omega) > t$ that $\tau_1(s, X^{t,x}_s(\omega))(\omega) > s$ for all $s \in (t, t_0(\omega))$. By (A.1) and (3.5), this in particular implies $(s, X^{t,x}_s(\omega)) \notin S_\tau$ for all $s \in (t, t_0(\omega))$. We therefore have

$$\begin{aligned} g(X^{t,x}_s(\omega)) &\le \mathbb{E}^{s, X^{t,x}_s(\omega)}\left[\delta(\mathcal{L}^*\tau(s, X_s) - s) g\left(X_{\mathcal{L}^*\tau(s,X_s)}\right)\right] \\ \text{(A.9)} \qquad &\le \mathbb{E}^{s, X^{t,x}_s(\omega)}\left[\delta(\mathcal{L}^*\tau(s, X_s) - [t_0]_{s,\omega}) g\left(X_{\mathcal{L}^*\tau(s,X_s)}\right)\right] \quad \forall s \in (t, t_0(\omega)), \end{aligned}$$

where the second line holds because δ is decreasing and δ, g are both nonnegative. On the other hand, by (2.2), it holds a.s. that

$$\text{(A.10)} \qquad \mathbb{E}[\delta(s_0 - t_0) g(X^{t,x}_{s_0}) \mid \mathcal{F}_s](\omega) = \mathbb{E}\left[\delta([s_0]_{s,\omega} - [t_0]_{s,\omega}) g([X^{t,x}_{s_0}]_{s,\omega})\right] \quad \forall s \ge t,\ s \in \mathbb{Q}.$$

Note that we have used the countability of $\mathbb{Q}$ to obtain the above almost-sure statement. For any fixed $\omega \in \Omega$ such that $t_0(\omega) \le s_0(\omega)$, if $s \in (t, t_0(\omega))$, then $s < s_0(\omega)$ and thus $[s_0]_{s,\omega}(\tilde{\omega}) = s_0(\omega \otimes_s \tilde{\omega}) = \mathcal{L}^*\tau(t,x)(\omega \otimes_s \tilde{\omega}) = \mathcal{L}^*\tau(s, X^{t,x}_s(\omega))(\tilde{\omega})$, for all $\tilde{\omega} \in \Omega$. We then conclude from (A.9) and (A.10) that it holds a.s. that

$$\text{(A.11)} \qquad g(X^{t,x}_s(\omega))\ 1_{\{(t,t_0(\omega))\cap\mathbb{Q}\}}(s) \le \mathbb{E}[\delta(s_0 - t_0) g(X^{t,x}_{s_0}) \mid \mathcal{F}_s](\omega)\ 1_{\{(t,t_0(\omega))\cap\mathbb{Q}\}}(s).$$

Since our sample space Ω is the canonical space equipped with the (augmented) natural filtration $\mathbb{F}$ generated by the Brownian motion W, the martingale representation theorem holds under current setting. This in particular implies that every martingale has a continuous version. Let $\{M_s\}_{s\ge t}$ be the continuous version of the martingale $\{\mathbb{E}[\delta(s_0 - t_0) g(X^{t,x}_{s_0}) \mid \mathcal{F}_s]\}_{s\ge t}$. Then, (A.11) immediately implies that it holds a.s. that

$$\text{(A.12)} \qquad g(X^{t,x}_s(\omega))\ 1_{\{(t,t_0(\omega))\cap\mathbb{Q}\}}(s) \le M_s(\omega)\ 1_{\{(t,t_0(\omega))\cap\mathbb{Q}\}}(s).$$

Also, using the right-continuity of M and (2.2), one can show that for any $\tau \in \mathcal{T}_t$, $M_\tau = \mathbb{E}[\delta(s_0 - t_0) g(X^{t,x}_{s_0}) \mid \mathcal{F}_\tau]$ a.s. Now, we can take some $\Omega^* \in \mathcal{F}_\infty$ with $\mathbb{P}[\Omega^*] = 1$ such that for all

$\omega \in \Omega^*$, $t_0(\omega) \le s_0(\omega)$, (A.12) holds true, and $M_{t_0}(\omega) = \mathbb{E}[\delta(s_0 - t_0) g(X^{t,x}_{s_0}) \mid \mathcal{F}_{t_0}](\omega)$. For any $\omega \in \Omega^* \cap \{t < t_0\}$, take $\{k_n\} \subset \mathbb{Q}$ such that $k_n > t$ and $k_n \uparrow t_0(\omega)$. Then, (A.12) implies

$$g(X^{t,x}_{k_n}(\omega)) \le M_{k_n}(\omega) \quad \forall n \in \mathbb{N}.$$

As $n \to \infty$, we obtain from the continuity of $s \mapsto X_s$ and $z \mapsto g(z)$, and the left-continuity of $s \mapsto M_s$ that $g(X^{t,x}_{t_0}(\omega)) \le M_{t_0}(\omega) = \mathbb{E}[\delta(s_0 - t_0) g(X^{t,x}_{s_0}) \mid \mathcal{F}_{t_0}](\omega)$. ◻

The previous lemma leads to useful properties of $\{\Theta^n \tau\}$ and $\{\mathcal{L}^* \Theta^n \tau\}$.

Proposition A.1. *Let Assumption 3.1 hold. If $\tau \in \mathcal{T}(\mathbb{X})$ satisfies Assumption 3.2, then for each $n \in \mathbb{N}$,*

(i) $\Theta^n \tau(t,x)(\omega) = t \implies \Theta^{n+1}\tau(t,x)(\omega) = t, \quad \forall (t,x,\omega) \in \mathbb{X} \times \Omega.$

(ii) $\mathcal{L}^* \Theta^{n+1}\tau(t,x)(\omega) \le \mathcal{L}^* \Theta^n \tau(t,x)(\omega), \quad \forall (t,x,\omega) \in \mathbb{X} \times \Omega.$

Proof. It suffices to prove the result for $n = 1$, as the remaining follows from induction in n. Fix $(t,x,\omega) \in \mathbb{X} \times \Omega$, and recall the notation in (A.1). We will first prove (i), i.e. $\tau_1(t,x)(\omega) = t$ implying $\tau_2(t,x)(\omega) = t$. Suppose $\tau_1(t,x)(\omega) = t$. If $\mathcal{L}^*\tau_1(t,x)(\omega) = t$, then (3.5) and Lemma A.1 immediately give $\tau_2(t,x)(\omega) = \Theta\tau_1(t,x)(\omega) = t$, as desired. We therefore assume below that $\mathcal{L}^*\tau_1(t,x)(\omega) > t$. By (3.5), $\tau_1(t,x)(\omega) = t$ implies

$$g(x) \ge \mathbb{E}[\delta(\mathcal{L}^*\tau(t,x) - t) g(X^{t,x}_{\mathcal{L}^*\tau(t,x)})]. \tag{A.13}$$

Let $t_0 := \mathcal{L}^*\tau_1(t,x)$ and $s_0 := \mathcal{L}^*\tau(t,x)$. Using the above inequality, $t_0 \le s_0$ a.s. (this is (3.8)), Assumption 3.1, and g being nonnegative, we obtain

$$\begin{aligned} g(x) &\ge \mathbb{E}[\delta(s_0 - t) g(X^{t,x}_{s_0})] \ge \mathbb{E}[\delta(t_0 - t)\delta(s_0 - t_0) g(X^{t,x}_{s_0})] \\ &= \mathbb{E}\left[\delta(t_0 - t)\mathbb{E}\left[\delta(s_0 - t_0) g(X^{t,x}_{s_0}) \mid \mathcal{F}_{t_0}\right]\right] \\ &\ge \mathbb{E}\left[\delta(t_0 - t) g(X^{t,x}_{t_0})\right], \end{aligned}$$

where the second line follows from the tower property of conditional expectations, and the third line is a consequence of Lemma A.2 (note that $t < t_0 \le s_0$ a.s. under current setting). This implies $(t,x) \notin C_{\tau_1}$, and thus

$$\tau_2(t,x)(\omega) = \begin{cases} t & \text{for } (t,x) \in S_{\tau_1} \\ \mathcal{L}\tau_1(t,x)(\omega) & \text{for } (t,x) \in I_{\tau_1} \end{cases} = t, \tag{A.14}$$

where $\mathcal{L}\tau_1(t,x)(\omega) = t$ because $\tau_1(t,x)(\omega) = t$. This finishes the proof of (i).

Now, observe that (ii) is a consequence of (i). If $\mathcal{L}^*\tau_1(t,x)(\omega) = t$, then there exist a sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$, depending on ω, such that $t_n \downarrow t$ and $\tau_1(t_n, X^{t,x}_{t_n}(\omega))(\omega) = t_n$ for all $n \in \mathbb{N}$. By (i), we must also have $\tau_2(t_n, X^{t,x}_{t_n}(\omega))(\omega) = t_n$ for all $n \in \mathbb{N}$. This implies $\mathcal{L}^*\tau_2(t,x)(\omega) = t = \mathcal{L}^*\tau_1(t,x)(\omega)$. On the other hand, if $t_0(\omega) := \mathcal{L}^*\tau_1(t,x)(\omega) > t$, then there are two possible cases: (1) $\mathcal{L}^*\tau_1(t_0(\omega), X^{t,x}_{t_0}(\omega))(\omega) > t_0(\omega)$. Then, the fact that $t < t_0(\omega) < \mathcal{L}^*\tau_1(t_0(\omega), X^{t,x}_{t_0}(\omega))(\omega)$ implies $\tau_1(t_0(\omega), X^{t,x}_{t_0}(\omega))(\omega) = t_0(\omega)$. By (i), we must also have $\tau_2(t_0(\omega), X^{t,x}_{t_0}(\omega))(\omega) = t_0(\omega)$. This implies $\mathcal{L}^*\tau_2(t,x)(\omega) \le t_0(\omega) = \mathcal{L}^*\tau_1(t,x)(\omega)$, as desired. (2) $\mathcal{L}^*\tau_1(t_0(\omega), X^{t,x}_{t_0}(\omega))(\omega) = t_0(\omega)$. Then, there exist a sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$, depending on ω, such that $t_n \downarrow t_0(\omega)$ and $\tau_1(t_n, X^{t,x}_{t_n}(\omega))(\omega) = t_n$ for all $n \in \mathbb{N}$. By (i), we must also have $\tau_2(t_n, X^{t,x}_{t_n}(\omega))(\omega) = t_n$ for all $n \in \mathbb{N}$. This implies $\mathcal{L}^*\tau_2(t,x)(\omega) \le t_0(\omega) = \mathcal{L}^*\tau_1(t,x)(\omega)$, as desired. ◻

Now, we are ready to prove Proposition 3.3.

Proof of Proposition 3.3. Fix $n \in \mathbb{N}$ and $(t,x,\omega) \in \mathbb{X} \times \Omega$. For a.e. $\omega \in \Omega$, if $\Theta^{n+1}\tau(t,x)(\omega) = t$, (3.9) holds trivially. Assume $\Theta^{n+1}\tau(t,x)(\omega) > t$. By (3.5) and Lemma A.1, this implies $\Theta^{n+1}\tau(t,x)(\omega) = \mathcal{L}^*\Theta^n\tau(t,x)(\omega) > t$. In view of Proposition A.1 (i), we must have $\Theta^n\tau(t,x)(\omega) > t$. We then conclude again from (3.5) and Lemma A.1 that $\Theta^n\tau(t,x)(\omega) = \mathcal{L}^*\Theta^{n-1}\tau(t,x)(\omega) > t$. Finally, we obtain from Proposition A.1 (ii) that $\Theta^{n+1}\tau(t,x)(\omega) = \mathcal{L}^*\Theta^n\tau(t,x)(\omega) \le \mathcal{L}^*\Theta^{n-1}\tau(t,x)(\omega) = \Theta^n\tau(t,x)(\omega)$.

Under (3.9), $\tau_0(t,x)$ in (3.6) is a well-defined limit of a sequence of stopping times in $\mathcal{T}_t$, and thus belongs to $\mathcal{T}_t$, too. Moreover, since $(t,x,\omega) \mapsto \Theta^n\tau(t,x)(\omega)$ is measurable for all $n \in \mathbb{N}$ (by Lemma 3.2), $(t,x,\omega) \mapsto \tau_0(t,x)(\omega) = \lim_{n\to\infty}\Theta^n\tau(t,x)(\omega)$ is measurable. This in particular implies that $\ker(\tau_0) = \{(t,x,\omega) \in \mathbb{X}\times\Omega : \tau_0(t,x)(\omega) = t\} \in \mathcal{B}(\mathbb{X}) \times \mathcal{F}_\infty$. We thus conclude that $\tau_0 \in \mathcal{T}(\mathbb{X})$. ☐

Remark A.1. *In general,* (3.9) *does not hold for* $n = 0$. *The proof above establishes* $\Theta^{n+1}\tau(t,x) \le \Theta^n\tau(t,x)$ *from* $\mathcal{L}^*\Theta^n\tau(t,x) \le \mathcal{L}^*\Theta^{n-1}\tau(t,x)$, *for all* $n \in \mathbb{N}$. *This argument, however, does not hold when* $n = 0$. *Thus, "*$\Theta\tau(t,x) \le \tau(t,x)$*" may not be true.*

A.4 Derivation of Theorem 3.1

Lemma A.3. *Suppose Assumption 3.1 holds and* $\tau \in \mathcal{T}(\mathbb{X})$ *satisfies* (3.8). *Given* $(t,x,\omega) \in \mathbb{X}\times\Omega$, τ_0 *defined in* (3.6) *satisfies*

$$\mathcal{L}^*\tau_0(t,x)(\omega) = \lim_{n\to\infty}\mathcal{L}^*\Theta^n\tau(t,x)(\omega).$$

Furthermore, on the set $\{\omega \in \Omega : \Theta^n\tau(t,x)(\omega) > t \text{ for all } n \in \mathbb{N}\}$, *we have*

$$\tau_0(t,x)(\omega) = \mathcal{L}^*\tau_0(t,x)(\omega) = \lim_{n\to\infty}\mathcal{L}^*\Theta^n\tau(t,x)(\omega).$$

Proof. We will write $\tau_n = \Theta^n\tau$ for all $n \in \mathbb{N}$, as in (A.1), and define $t_0 := \lim_{n\to\infty}\mathcal{L}^*\tau_n(t,x)$.
Case I: $\mathcal{L}^*\tau_\ell(t,x)(\omega) = t$ for some $\ell \in \mathbb{N}$. By Proposition A.1 (ii), we have $t_0(\omega) = t$. By (3.3), there exist a sequence $\{t_m\}_{m\in\mathbb{N}}$ in $\mathbb{R}_+$, depending on ω, such that $t_m \downarrow t$ and $\tau_\ell(t_m, X^{t,x}_{t_m}(\omega))(\omega) = t_m$ for all $m \in \mathbb{N}$. By Theorem 3.3, this implies that $\tau_0(t_m, X^{t,x}_{t_m}(\omega))(\omega) = t_m$ for all $m \in \mathbb{N}$. We thus conclude that $\mathcal{L}^*\tau_0(t,x)(\omega) = t = t_0(\omega)$.
Case II: $\mathcal{L}^*\tau_n(t,x)(\omega) > t$ for all $n \in \mathbb{N}$ and $t_0(\omega) = t$. Let $t_n := \mathcal{L}^*\tau_n(t,x)(\omega) > t$. For each $n \in \mathbb{N}$, in view of Lemma A.1, we have either (i) $\tau_n(t_n, X^{t,x}_{t_n}(\omega))(\omega) = t_n$ or (ii) there exist a sequence $\{s_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}_+$, depending on ω, such that $s_k \downarrow t_n$ and $\tau_n(s_k, X^{t,x}_{s_k}(\omega))(\omega) = s_k$ for all $k \in \mathbb{N}$. Since $t_n \downarrow t_0(\omega) = t$, we can construct from (i) and (ii) a real sequence $\{\ell_n\}_{n\in\mathbb{N}}$ such that $\ell_n \downarrow t$ and $\tau_n(\ell_n, X^{t,x}_{\ell_n}(\omega))(\omega) = \ell_n$ for all $n \in \mathbb{N}$. By Theorem 3.3, this implies $\tau_0(\ell_n, X^{t,x}_{\ell_n}(\omega))(\omega) = \ell_n$ for all $n \in \mathbb{N}$. We thus conclude that $\mathcal{L}^*\tau_0(t,x)(\omega) = t = t_0(\omega)$.
Case III: $\mathcal{L}^*\tau_n(t,x)(\omega) > t$ for all $n \in \mathbb{N}$ and $t_0(\omega) > t$. By Proposition A.1 (ii), we must have $\mathcal{L}^*\tau_n(t,x)(\omega) \ge t_0(\omega) > t$ for all $n \in \mathbb{N}$. This implies that for all $s \in (t, t_0(\omega))$ and $n \in \mathbb{N}$,

$$\mathcal{L}^*\tau_n(s, X^{t,x}_s(\omega))(\omega) = \mathcal{L}^*\tau_n(t,x)(\omega), \tag{A.15}$$
$$\tau_n(s, X^{t,x}_s(\omega))(\omega) > s. \tag{A.16}$$

In view of (3.5) and Lemma A.1, (A.16) implies

$$\tau_{n+1}(s, X^{t,x}_s(\omega))(\omega) = \Theta\tau_n(s, X^{t,x}_s(\omega))(\omega) = \mathcal{L}^*\tau_n(s, X^{t,x}_s(\omega))(\omega). \tag{A.17}$$

Thanks to (A.17) and (A.15), we get

$$\begin{aligned}\tau_0(s, X^{t,x}_s(\omega))(\omega) &= \lim_{n\to\infty}\tau_n(s, X^{t,x}_s(\omega))(\omega) = \lim_{n\to\infty}\mathcal{L}^*\tau_n(s, X^{t,x}_s(\omega))(\omega)\\ &= \lim_{n\to\infty}\mathcal{L}^*\tau_n(t,x)(\omega) = t_0(\omega) \quad \forall s \in (t, t_0(\omega)).\end{aligned} \tag{A.18}$$

This in particular shows that

$$\tau_0(s, X_s^{t,x}(\omega))(\omega) > s \quad \forall s \in (t, t_0(\omega)). \tag{A.19}$$

To complete the proof, we deal with two sub-cases.

> **Case III-1:** $\mathcal{L}^*\tau_n(t,x)(\omega) > t_0(\omega)$ for all $n \in \mathbb{N}$. Then, we observe that (A.15), (A.16), and (A.17) remain true with $s = t_0(\omega)$. It follows that (A.18) also holds with $s = t_0(\omega)$. This, together with (A.19), shows that $\mathcal{L}^*\tau_0(t,x)(\omega) = t_0(\omega)$.
>
> **Case III-2:** $\mathcal{L}^*\tau_\ell(t,x)(\omega) = t_0(\omega)$ for some $\ell \in \mathbb{N}$. If $\tau_\ell(t_0(\omega), X_{t_0}^{t,x}(\omega))(\omega) = t_0(\omega)$, by Theorem 3.3 we have $\tau_0(t_0(\omega), X_{t_0}^{t,x}(\omega))(\omega) = t_0(\omega)$. This, together with (A.19), shows that $\mathcal{L}^*\tau_0(t,x)(\omega) = t_0(\omega)$. On the other hand, if $\tau_\ell(t_0(\omega), X_{t_0}^{t,x}(\omega)) > t_0(\omega)$, we see from Lemma A.1 that there exist $\{t_m\}_{m\in\mathbb{N}}$ in $\mathbb{R}_+$, depending on ω, such that $t_m \downarrow t_0(\omega)$ and $\tau_\ell(t_m, X_{t_m}^{t,x}(\omega))(\omega) = t_m$ for all $m \in \mathbb{N}$. By Theorem 3.3 again, $\tau_0(t_m, X_{t_m}^{t,x}(\omega))(\omega) = t_m$ for all $m \in \mathbb{N}$. This implies $\mathcal{L}^*\tau_0(t,x)(\omega) \le t_m$ for all $m \in \mathbb{N}$, and thus $\mathcal{L}^*\tau_0(t,x)(\omega) \le t_0(\omega)$. Since (A.19) already implies $\mathcal{L}^*\tau_0(t,x)(\omega) \ge t_0(\omega)$, we conclude $\mathcal{L}^*\tau_0(t,x)(\omega) = t_0(\omega)$.

Finally, on the set $\{\omega \in \Omega : \tau_n(t,x)(\omega) > t \text{ for all } n \in \mathbb{N}\}$, we observe from (3.5) and Lemma A.1 that $\tau_{n+1}(t,x) = \mathcal{L}^*\tau_n(t,x)$ for all $n \in \mathbb{N}$. It follows that

$$\tau_0(t,x) = \lim_{n\to\infty} \tau_n(t,x) = \lim_{n\to\infty} \mathcal{L}^*\tau_n(t,x). \tag{A.20}$$

□

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Theorem 3.3, $\tau_0 \in \mathcal{T}(\mathbb{X})$ is well-defined. For simplicity, we will write $\tau_n = \Theta^n\tau$ for all $n \in \mathbb{N}$. Fix $(t,x,\omega) \in \mathbb{X} \times \Omega$. We first deal with the case where $\tau_\ell(t,x)(\omega) = t$ for some $\ell \in \mathbb{N}$. By Proposition A.1 (i), $\tau_n(t,x)(\omega) = t$ for $n \ge \ell$. This implies $\tau_0(t,x)(\omega) = t$ and thus $\mathcal{L}\tau_0(t,x) = t$. Observe from (3.5) and $\tau_n(t,x)(\omega) = t$ $\forall n \ge \ell$ that $(t,x) \notin C_{\tau_n}$ $\forall n \ge \ell$, i.e.

$$g(x) \ge \mathbb{E}^{t,x}\left[\delta(\mathcal{L}^*\tau_n(t,x) - t) g(X_{\mathcal{L}^*\tau_n(t,x)})\right] \quad \forall n \ge \ell.$$

Sending $n \to \infty$, we obtain from Lemma A.3 below that

$$g(x) \ge \mathbb{E}^{t,x}\left[\delta(\mathcal{L}^*\tau_0(t,x) - t) g(X_{\mathcal{L}^*\tau_0(t,x)})\right],$$

which shows that $(t,x) \notin C_{\tau_0}$. We then conclude from (3.5) and $\mathcal{L}\tau_0(t,x) = t$ that $\Theta\tau_0(t,x)(\omega) = t = \tau_0(t,x)(\omega)$. For the other case where $\tau_n(t,x)(\omega) > t$ for all $n \in \mathbb{N}$, we have $\tau_{n+1}(t,x) = \Theta\tau_n(t,x) > t$ for all $n \in \mathbb{N}$. This, together with (3.5), implies $(t,x) \notin S_{\tau_n}$, i.e.

$$g(x) \le \mathbb{E}^{t,x}\left[\delta(\mathcal{L}^*\tau_n(t,x) - t) g(X_{\mathcal{L}^*\tau_n(t,x)})\right] \quad \forall n \in \mathbb{N}.$$

Sending $n \to \infty$, we obtain from Lemma A.3 that

$$g(x) \le \mathbb{E}^{t,x}\left[\delta(\mathcal{L}^*\tau_0(t,x) - t) g(X_{\mathcal{L}^*\tau_0(t,x)})\right],$$

which shows that $(t,x) \notin S_{\tau_0}$. In view of (3.5), Lemma A.1, and Lemma A.3, we have $\Theta\tau_0(t,x)(\omega) = \mathcal{L}^*\tau_0(t,x)(\omega) = \tau_0(t,x)(\omega)$. We therefore conclude that $\tau_0 \in \mathcal{E}(\mathbb{X})$. □

B Proofs for Section 4

B.1 Derivation of Proposition 4.1

An unusual feature of (4.2) is that the discounted process $\{\delta(s-t)v(s,X_s^{t,x})\}_{s\geq t}$ may not be a supermartingale. To see this, observe that in the classical case of exponential discounting, (2.12) ensures

$$\delta(s-t)v(s,X_s^{t,x}) = \sup_{\tau\in\mathcal{T}_s} \mathbb{E}^{s,X_s^{t,x}}\left[\delta(s-t)\delta(\tau-s)g(X_\tau)\right] = \sup_{\tau\in\mathcal{T}_s} \mathbb{E}^{s,X_s^{t,x}}\left[\delta(\tau-t)g(X_\tau)\right] \quad s\geq t,$$

which shows that $\{\delta(s-t)v(s,X_s^{t,x})\}_{s\geq t}$ is a supermartingale. Under hyperbolic discounting (4.1), since $\delta(r_1)\delta(r_2)<\delta(r_1+r_2)$ for all $r_1,r_2\geq 0$, $\{\delta(s-t)v(s,X_s^{t,x})\}_{s\geq t}$ no longer coincides with the supermatingale $\{\sup_{\tau\in\mathcal{T}_s}\mathbb{E}^{s,X_s}[\delta(\tau-t)g(X_\tau)]\}_{s\geq t}$. Standard approaches for optimal stopping problems, therefore, can not be applied to (4.2).

To overcome this, we introduce an auxiliary value function. For any fixed $t\geq 0$, define

$$\begin{aligned} V(t,s,x) &:= \sup_{\tau\in\mathcal{T}_s}\mathbb{E}^{s,x}\left[\delta(\tau-t)g(X_\tau)\right] \\ &= \sup_{\tau\in\mathcal{T}_s}\mathbb{E}^{s,x}\left[\frac{|X_\tau|}{1+\beta(\tau-t)}\right], \quad \text{for } (s,x)\in[t,\infty)\times\mathbb{R}. \end{aligned} \tag{B.1}$$

By definition, $V(t,t,x)=v(t,x)$, and the process $\{V(t,s,X_s^{t,x})\}_{s\geq t}$ is a supermartingale. The associated variational inequality for $V(t,\cdot,\cdot)$ is the following: for $(s,x)\in[t,\infty)\times\mathbb{R}$,

$$\min\left\{w_s(t,s,x)+\frac{1}{2}w_{xx}(t,s,x),\ w(t,s,x)-\frac{|x|}{1+\beta(s-t)}\right\}=0. \tag{B.2}$$

We are able to solve this equation explicitly.

Proof of Proposition 4.1. The equation (B.2) can be rewritten as a free boundary problem

$$\begin{cases} v_s(t,s,x)+\frac{1}{2}v_{xx}(t,s,x)=0, \quad v(t,s,x)>\frac{|x|}{1+\beta(s-t)}, & \text{for } |x|<b(t,s);\\ v(t,s,x)=\frac{|x|}{1+\beta(s-t)}, & \text{for } |x|\geq b(t,s). \end{cases} \tag{B.3}$$

where $s\mapsto b(t,s)$ is the free boundary to be determined. Following Pederson and Perkir (2000), we propose the ansatz

$$w(t,s,x)=\frac{1}{\sqrt{1+\beta(s-t)}}h\left(\frac{x}{\sqrt{1+\beta(s-t)}}\right).$$

Equation (B.3) then becomes a one-dimensional free boundary problem:

$$\begin{cases} -\beta y h'(y)+h''(y)=\beta h(y), \quad h(y)>|y|, & \text{for } |y|<\frac{b(t,s)}{\sqrt{1+\beta(s-t)}};\\ h(y)=|y|, & \text{for } |y|\geq\frac{b(t,s)}{\sqrt{1+\beta(s-t)}}. \end{cases} \tag{B.4}$$

Since (t,s) does not appear in the above ODE, we take $b(t,s)=\alpha\sqrt{1+\beta(s-t)}$ for some $\alpha\geq 0$. The general solution of the first line of (B.4) is

$$h(y)=e^{\frac{\beta}{2}y^2}\left(c_1+c_2\sqrt{\frac{2}{\beta}}\int_0^{\sqrt{\beta/2}y}e^{-u^2}du\right), \quad (c_1,c_2)\in\mathbb{R}^2\,.$$

The second line of (B.4) gives $h(\alpha)=\alpha$. We then have

$$w(t,s,x)=\begin{cases}\frac{e^{\frac{\beta x^2}{2(1+\beta(s-t))}}}{\sqrt{1+\beta(s-t)}}\left(c_1+c_2\sqrt{\frac{2}{\beta}}\int_0^{\frac{\sqrt{\beta/2}x}{\sqrt{1+\beta(s-t)}}}e^{-u^2}du\right), & |x|<\alpha\sqrt{1+\beta(s-t)};\\ \frac{|x|}{1+\beta(s-t)}, & |x|\geq\alpha\sqrt{1+\beta(s-t)}.\end{cases}$$

To find the parameters c_1, c_2 and α, we equate the partial derivatives of $(s,x)\mapsto w(t,s,x)$ obtained on both sides of the free boundary. This yields the equations

$$\alpha=e^{\frac{\beta}{2}\alpha^2}\left(c_1+c_2\sqrt{\frac{2}{\beta}}\int_0^{\sqrt{\beta/2}\alpha}e^{-u^2}du\right)\quad\text{and}\quad \mathrm{sgn}(x)-c_2=\mathrm{sgn}(x)\alpha^2\beta.$$

The last equation implies $c_2=0$. This, together with the first equation, shows that $\alpha=1/\sqrt{\beta}$ and $c_1=\alpha e^{-1/2}$. Thus, we obtain

$$\text{(B.5)}\qquad w(t,s,x)=\begin{cases}\frac{1}{\sqrt{\beta}\sqrt{1+\beta(s-t)}}\exp\left(\frac{1}{2}\left(\frac{\beta x^2}{1+\beta(s-t)}-1\right)\right) & \text{for } |x|<\sqrt{1/\beta+(t-s)},\\ w(t,s,x)=\frac{|x|}{1+\beta(s-t)} & \text{for } |x|\geq\sqrt{1/\beta+(t-s)}.\end{cases}$$

Notice that $w(t,\cdot,\cdot)$ is $\mathcal{C}^{1,1}$ on $[t,+\infty)\times\mathbb{R}$, and $\mathcal{C}^{1,2}$ on the domain

$$\left\{(s,x)\in[t,\infty)\times\mathbb{R}\ :\ |x|<\sqrt{1/\beta+(t-s)}\right\}.$$

Moreover, from (B.5), $w_s(t,s,x)+\frac{1}{2}w_{xx}(t,s,x)<0$ for $|x|>\sqrt{1/\beta+(s-t)}$. We therefore conclude from the standard verification theorem (see e.g. Øksendal and Sulem (2007, Theorem 3.2)) that $V(t,s,x)=w(t,s,x)$ is a smooth solution of (B.3). This implies that $\{V(t,s,X_s^{t,x})\}_{s\geq t}$ is a supermartingale, and $\{V(t,s\wedge\tau^*(t,x),X^{t,x}_{s\wedge\tau^*(t,x)})\}_{s\geq t}$ is a true martingale, with $\tau^*(t,x):=\inf\{s\geq t:|x|\geq\sqrt{1/\beta+(t-s)}\}$. It then follows from standard arguments that $\tau^*(t,x)$ is the smallest optimal stopping time of (4.2). In view of Proposition 2.1, we must have $\widetilde{\tau}(t,x)=\tau^*(t,x)$. $\square$

B.2 Proof of Lemma 4.1

Given $\tau\in\mathcal{E}'(\mathbb{X})=\mathcal{T}'(\mathbb{X})\cap\mathcal{E}(\mathbb{X})$, we have

$$\text{(B.6)}\qquad \Theta(t,x)=\tau(t,x)=\inf\{s\geq t:|X_s^{t,x}|\in L\}\quad\text{for all } (t,x)\in\mathbb{X},$$

for some Borel subset L of $\mathbb{R}_+$. Set $a:=\inf L$. We claim that $\tau(t,x)=\tau_a(t,x)=\inf\{s\geq t: |X_s^{t,x}|\geq a\}$ for all $(t,x)\in\mathbb{X}$. If $L=[a,\infty)$, there is nothing to prove. Assume that there exists $x\geq a$ such that $x\notin L$. Define

$$\text{(B.7)}\qquad \ell:=\sup\{a\in L\ :\ a<x\}\quad\text{and}\quad u:=\inf\{b\in L\ :\ b>x\}.$$

Case I: $x=a$. As $a=\inf L$, there exist $\{a_n\}_{n\in\mathbb{N}}$ in L such that $a_n\downarrow a$. Thus, by the same calculation as in (4.9), we have $|x|>J(t,x;\mathcal{L}^*\tau(t,x))$. By (3.5), this implies $\Theta\tau(t,x)(\omega)=t$, and thus $\tau(t,x)(\omega)=t$, for all $\omega\in\Omega$. In view of (B.6) and X being a Brownian motion, we must have $x\in L$, a contradiction.
Case II: $x>a$ and $\ell<u$. Take $y\in(\ell,u)$. Then $\tau(t,y)>t$, and thus $\Theta\tau(t,y)>t$. By (3.5) and Lemma A.1, this implies $\Theta\tau(t,y)=\mathcal{L}^*\tau(t,y)$. We therefore have $\tau(t,y)=\mathcal{L}^*\tau(t,y)$. Now,

$$J(t,y;\mathcal{L}^*\tau(t,y))=J(t,y;\tau(t,y))<\mathbb{E}^{t,y}[|X_{\tau(t,y)}|]=\ell\mathbb{P}[|X_{\tau(t,y)}|=\ell]+u\mathbb{P}[|X_{\tau(t,y)}|=u]=x,$$

where the last equality follows from $\mathbb{P}[|X_{\tau(t,y)}|=\ell]=\frac{u-y}{u-\ell}$ and $\mathbb{P}[|X_{\tau(t,y)}|=u]=\frac{y-\ell}{u-\ell}$, obtained from the optional sampling theorem. By (3.5), this implies $\Theta\tau(t,y)=t$, a contradiction.
Case II: $x>a$ and $\ell=u=x$. Since $x>a$, we trivially have $\tau_a(t,x)=t$. On the other hand, $\ell=u=x$ implies $\mathcal{L}^*\tau(t,x)=t$. By (3.5), we have $\Theta\tau(t,x)=t$, and thus $\tau(t,x)=t$. We therefore get $\tau(t,x)=\tau_a(t,x)$.

B.3 Proof of Lemma 4.2

(i) Given $a \geq 0$, it is obvious from definition that $\eta(0,a) \in (0,a)$ and $\eta(a,a) = a$. Fix $x \in (0,a)$, and let f_a^x denote the density of $T_a^x := \tau_a(t,x) - t$. We obtain

$$
\begin{aligned}
\mathbb{E}_{t,x}\left[\frac{1}{1+\beta T_a^x}\right] &= \int_0^\infty \frac{1}{1+\beta t} f_a^x(t)dt = \int_0^\infty \int_0^\infty e^{-(1+\beta t)s} f_a^x(t)ds\ dt \\
&= \int_0^\infty e^{-s}\left(\int_0^\infty e^{-\beta st} f_a^x(t)dt\right)\, ds = \int_0^\infty e^{-s}\mathbb{E}_{t,x}[e^{-\beta s T_a^x}]ds.
\end{aligned} \tag{B.8}
$$

Notice that T_a^x is the first hitting time to a by an one-dimensional Bessel process. We thus compute its Laplace transform from the formula in Brenner and Scott (2002):

$$
\mathbb{E}_{t,x}\left[e^{-\frac{\lambda^2}{2}T_a^x}\right] = \frac{\sqrt{x} I_{-\frac{1}{2}}(x\lambda)}{\sqrt{a} I_{-\frac{1}{2}}(a\lambda)} = \cosh(x\lambda)\operatorname{sech}(a\lambda), \qquad \text{for } x \leq a. \tag{B.9}
$$

Here, I_ν denotes the modified Bessel function of the first kind. Thanks to the above formula with $\lambda = \sqrt{2\beta s}$, we obtain from (B.8) that

$$
\eta(x,a) = a\int_0^\infty e^{-s}\cosh(x\sqrt{2\beta s})\operatorname{sech}(a\sqrt{2\beta s})ds. \tag{B.10}
$$

It is then obvious that $x \mapsto \eta(x,a)$ is strictly increasing. Moreover,

$$
\eta_{xx}(x,a) = 2a\beta^2\int_0^\infty e^{-s} s\cosh(x\sqrt{2\beta s})\operatorname{sech}(a\sqrt{2\beta s})ds > 0 \quad \text{ for } x \in [0,a],
$$

which shows the strict convexity.

(ii) This follows from (B.10) and the dominated convergence theorem.

(iii) The proof divides into two parts. We will first prove the desired result with $x^*(a) \in (0,a)$, and then upgrade it to $x^*(a) \in (0,a^*)$ in the second step. Fix $a \geq 0$. In view of the properties in part (i), we observe that the two curves $y = \eta(x,a)$ and $y = x$ intersect at some $x^*(a) \in (0,a)$ if and only if $\eta_x(a,a) > 1$. Define $k(a) := \eta_x(a,a)$. By (B.10),

$$
k(a) = a\int_0^\infty e^{-s}\sqrt{2\beta s}\tanh(a\sqrt{2\beta s})ds.
$$

Thus, we see that $k(0) = 0$ and $k(a)$ is strictly increasing on $(0,1)$, since for any $a > 0$,

$$
k'(a) = \int_0^\infty e^{-s}\sqrt{2s}\left(\tanh(a\sqrt{2s}) + \frac{a\sqrt{2s}}{\cosh^2(a\sqrt{2s})}\right) ds > 0.
$$

By numerical computation, we find

$$
k(1/\sqrt{\beta}) = \int_0^\infty e^{-s}\sqrt{2s}\tanh(\sqrt{2s})ds \approx 1.07461 > 1.
$$

It follows that there must exist $a^* \in (0, 1/\sqrt{\beta})$ such that $k(a^*) = \eta_x(a^*,a^*) = 1$. Monotonicity of $k(a)$ then gives the desired result.

Now, for any $a > a^*$, we intend to upgrade the previous result to $x^*(a) \in (0,a^*)$. Fix $x \geq 0$. By the definition of η and part (ii), on the domain $a \in [x,\infty)$, the map $a \mapsto \eta(x,a)$ must either first increases and then decreases to 0, or directly decreases down to 0. From (B.10), we have

$$
\eta_a(x,x) = 1 - x\int_0^\infty e^{-s}\sqrt{2\beta s}\tanh(x\sqrt{2\beta s})ds = 1 - k(x),
$$

where k is defined in part (iii). Recalling that $k(a^*) = 1$, we have $\eta_a(a^*, a^*) = 0$. Notice that

$$\begin{aligned}\eta_{aa}(a^*, a^*) &= -\frac{2}{a^*}k(a^*) - 2\beta a^* + a^* \int_0^\infty 4\beta s e^{-s} \tanh^2(a^* \sqrt{2\beta s}) ds \\ &\leq -\frac{2}{a^*} + 2\beta a^* < 0,\end{aligned}$$

where the second line follows from $\tanh(x) \leq 1$ for $x \geq 0$ and $a^* \in (0, 1/\sqrt{\beta})$. Since $\eta_a(a^*, a^*) = 0$ and $\eta_{aa}(a^*, a^*) < 0$, we conclude that on the domain $a \in [a^*, \infty)$, the map $a \mapsto \eta(a^*, a)$ decreases down to 0. Now, for any $a > a^*$, since $\eta(a^*, a) < \eta(a^*, a^*) = a^*$, we must have $x^*(a) < a^*$.

B.4 Proof of Lemma 4.3

Fix $0 \leq x < a$ and $t \geq 0$. Set $T_a^x := \tau_a(t, x) - t$ and $\hat{T}_a^x := \tau_a'(t, x) - t$. Since $\tau_a \leq \tau_a' \leq \tau_{a+\varepsilon}$ by definition for all $\varepsilon > 0$, (B.9) implies that for any $\lambda > 0$,

$$\begin{aligned}\cosh(\lambda x)\operatorname{sech}(\lambda(a+\varepsilon)) = \mathbb{E}_{t,x}\left[e^{-\frac{\lambda^2}{2}T_{a+\varepsilon}^x}\right] &\leq \mathbb{E}_{t,x}\left[e^{-\frac{\lambda^2}{2}\hat{T}_a^x}\right] \\ &\leq \mathbb{E}_{t,x}\left[e^{-\frac{\lambda^2}{2}T_a^x}\right] = \cosh(\lambda x)\operatorname{sech}(\lambda a), \quad \forall \varepsilon > 0.\end{aligned}$$

As $\varepsilon \to \infty$, we obtain $\mathbb{E}_{t,x}[e^{-\frac{\lambda^2}{2}\hat{T}_a^x}] = \mathbb{E}_{t,x}[e^{-\frac{\lambda^2}{2}T_a^x}]$. This, together with $T_a^x \leq \hat{T}_a^x$, implies $T_a^x = \hat{T}_a^x$ a.s., or $\tau_a(t, x) = \tau_a'(t, x)$ a.s.

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